Constructive Combinatorics — So 2018
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Exercise Sheet 2

Due date: May 9th, 12:30 PM

You should try to solve and write clear solutions to as many of the exercises as you can.

Exercise 1 A *projective plane* Π is a hypergraph with vertex set $\mathcal{P}$ and edge set $\mathcal{L} \subseteq 2^{\mathcal{P}}$, having the following properties:

(i) $\forall \ell_1, \ell_2 \in \mathcal{L}$, $\ell_1 \neq \ell_2$, we have $|\ell_1 \cap \ell_2| = 1$

(ii) $\forall p_1, p_2 \in \mathcal{P}, p_1 \neq p_2, \; \exists! \ell \in \mathcal{L}$ with $p_1, p_2 \in \ell$

(iii) $\exists p_1, p_2, p_3, p_4 \in \mathcal{P}$ such that $\forall \ell \in \mathcal{L}$ we have $|\ell \cap \{p_1, p_2, p_3, p_4\}| \leq 2$.

The elements of $\mathcal{P}$ are called *point*s, and the elements of $\mathcal{L}$ are called *line*s. When $p \in \ell \in \mathcal{L}$ then we also say that point p is *incident to* line ℓ. The first two properties require that every pair of distinct lines has a unique common point, every pair of distinct points is incident to a unique line. The third property excludes degenerate situations, when among any four points three are collinear.

A projective plane is called finite if $\mathcal{P}$ is a finite set. Prove that for any finite projective plane there is an integer $m \geq 2$ such that every point (line, respectively) of Π is incident to exactly $m+1$ lines (points, respectively) and that $|\mathcal{P}| = m^2 + m + 1 = |\mathcal{L}|$. (The integer m is called the *order* of the finite projective plane.)

Exercise 2 A particular example of a projective plane is $PG(2, q)$, constructed with the help of the q-element field $\mathbb{F}_q$.

The set of points of $PG(2, q)$ is defined as

$$\mathcal{P} := \left\{ [x_0, x_1, x_2] : (x_0, x_1, x_1) \in \mathbb{F}_q^3 \setminus \{(0,0,0)\} \right\},$$

where

$$[x_0, x_1, x_2] := \left\{ (cx_0, cx_1, cx_2) \;:\; c \in \mathbb{F}_q^* \right\}.$$

The set of lines of $PG(2, q)$ is defined as

$$\mathcal{L} := \left\{ L(a_0, a_1, a_2) : (a_0, a_1, a_2) \in \mathbb{F}_q^3 \setminus \{(0,0,0)\} \right\},$$

where for a triple $(a_0, a_1, a_2) \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}$ the corresponding line is defined as

$$L(a_0, a_1, a_2) := \{ [x_0, x_1, x_2] \in \mathcal{P} \;:\; a_0 x_0 + a_1 x_1 + a_2 x_2 = 0 \}.$$

Show that the hypergraph is well-defined and $PG(2, q)$ is a projective plane of order q (in the sense of Exercise 1).

Exercise 3 For a prime p and every $\alpha \in \mathbb{F}_p$, determine the number of solutions in $(x, y) \in \mathbb{F}_p^2$ to the equation $x^2 + y^2 = \alpha$.

[Hint (to be read backwards): tI yam pleh ot tnemirepxe htiw llams seulav fo p dna α.]

Exercise 4 Given a bipartite graph H, define $\mathrm{ex}(n, m, H)$ as the maximum number of edges in an H-free bipartite graph with partite sets of size n and m respectively. Show that $\mathrm{ex}(q^2 + q + 1, q^2 + q + 1, K_{2,2}) = (q^2 + q + 1)(q + 1)$ for every prime power q.

Exercise 5 For a prime p and every $\alpha \in \mathbb{F}_p$, determine the number of solutions $(x, y, z) \in \mathbb{F}_p^3$ to the equation $x^2 + y^2 + z^2 = \alpha$. Determine the exact number of edges of the polarity graph.