Constructive Combinatorics So 2018
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Exercise Sheet 3

Due date: May 28th, 10:30AM.

You should try to solve and write clear solutions to as many of the exercises as you can.

Exercise 1 Give a general exact formula for the number of solutions to $x_1^2 + \cdots + x_k^2 = \beta$, for any fixed $k \in \mathbb{N}, \beta \in \mathbb{F}_p$.

Exercise 2 This exercise shows that a straightforward generalisation of the $K_{2,s}$-free constructions we have seen fails to provide $K_{3,s}$-free graphs.

A *hyperplane* in $\mathbb{F}_q^3$ is an affine subspace of dimension two. We construct a bipartite graph G, with vertex classes $\mathcal{P} = \mathbb{F}_q^3$ and $\mathcal{H}$, the set of hyperplanes in $\mathbb{F}_q^3$. A pair $(p, H) \in \mathcal{P} \times \mathcal{H}$ forms an edge if and only if $p \in H$. (This is the generalisation of Construction 0 for $K_{2,2}$-free graphs.)

(a) How many vertices and edges does G have?

(b) Show that if we choose H_1, H_2, and H_3 uniformly at random from $\mathcal{H}$, the probability that $|H_1 \cap H_2 \cap H_3| = 1$ tends to 1 as $q \to \infty$.

(b) Despite this fact, show that any choice of $10q^2$ hyperplanes contains three planes that intersect in at least q points, thus inducing a $K_{3,q}$ in G.

Exercise 3 Show that if $-\alpha \in QR(p)$, then the α-Brown graph contains not only a $K_{3,3}$, but also a $K_{(1+o(1))n^{1/3},(1+o(1))n^{1/3}}$. Can you also find a $K_{n^{1/3},n^{2/3}}$?

Exercise 4 Prove that in the α-Brown graph, when $-\alpha \notin QR(p)$, roughly half of the triples of vertices have two common neighbours while the other half have none. Even further, describe explicitely most of the triples without common neighbours.

Exercise 5

(a) Improve the KST-upper-bound a bit. Show that for arbirary $s \geq 3$, we have $ex(n, K_{3,s}) \lessapprox \frac{\sqrt[3]{s-2}}{2} n^{5/3}$.

(b) Generalize the proof above and show that $ex(n, K_{4,4}) \lessapprox \frac{1}{2} n^{7/4}$. (Hint: Instead of lower bounding $\sum \binom{x_i}{3}$ in terms of $(\sum x_i)^3$ (which follows from the convexity of $\binom{x}{3}$) you might want to bound it from below in terms of the product of $\sum \binom{x_i}{2}$ and $\sum x_i$.)

Exercise 6 A couple of failed constructions.

(a) A natural thought to extend the idea of the Brown graph to $K_{4,4}-$ or $K_{4,1000}-$avoiding dense graphs is the following. Instead of three dimensions let us take four, i.e., our vertex set is $\mathbb{F}_p^4$. Let the neighborhood of a vertex x be determined by a four-dimensional sphere around it, in particular a vertex y is adjacent to x if and only if $\sum_i (y_i - x_i)^2 = 1$. According to Exercise 1, our graph has roughly $cn^{7/4}$ edges — the conjectured truth. Prove, however, that this graph contains a $K_{(1+o(1)n^{1/4},(1+o(1))n^{1/4}}$.

(b) Prove that even taking a higher degree surface of the form $\sum_i (y_i - x_i)^{1000} = 1$ as the neighborhood of x, instead of the sphere, would not help us. (Note that the theorem of Schmidt we proved in class ensures that this graph also has roughly the correct number $cn^{7/4}$ of edges.)

(c) Let the vertex set of a graph G be $\mathbb{F}_p^4$. Let (a, b, c, d) be adjacent to (a', b', c', d') if and only if $(a+a')(b+b')(c+c')(d+d') = 1$. Prove that G contains a $K_{(1+o(1))n^{1/4},(1+o(1))n^{1/4}}$.