

Exercise Sheet 4

Due date: June 6th 12:30 PM

You should try to solve and write clear solutions to as many of the exercises as you can.

There are two left-over exercises from the last sheet: Exercise 4 and 5. These will enjoy preference at the discussion. Here are four more to ponder.

Exercise 1 The k -colour Ramsey number $R_k(G)$ is the smallest integer n such that in every colouring of the edges of K_n with k colours, there is a monochromatic copy of G .

(a) (5 points) Prove (with the probabilistic method) that $R_k(K_{3,3}) = \Omega((k/\log k)^3)$ (where the exponent 3 represents the reciprocal of 2 minus the exponent of the Turán number of $K_{3,3}$).

(b) (10 points) Show that $R_k(K_{3,3}) = (1 + o(1))k^3$.

[Hint: [to be read backwards] roF eht rewol dnuob, esu eht evitcejorp shparg-mron dna eht yeK ammeL.]

Exercise 2 Prove the even-Moore bound: that for any n -vertex d -regular graph G with girth $g(G) \geq 2k$,

$$n \geq 2 \sum_{i=0}^{k-1} (d-1)^i.$$

Can you extend this to irregular graphs?

Exercise 3 Let X and Y be discrete random variables. Let \mathbb{E}_x denote the expectation when x is distributed as X , and \mathbb{E}_y the expectation when y is distributed as Y . We define the *entropy* of X as $H(X) = \mathbb{E}_x[-\log_2 \mathbb{P}(X = x)]$,¹ where we set $z \log_2 z = 0$ for $z = 0$. We can also define the *joint entropy* $H(X, Y) = \mathbb{E}_{x,y}[-\log_2 \mathbb{P}(X = x, Y = y)]$ and the *conditional entropy* $H(X|Y) = \mathbb{E}_y[\mathbb{E}_{x|y}[-\log_2 \mathbb{P}(X = x|Y = y)]]$. Prove that $H(X, Y) = H(Y) + H(X|Y)$.

¹By definition of \mathbb{E}_x , this is equivalent to $-\sum_x \mathbb{P}(X = x) \log_2 \mathbb{P}(X = x)$.

Exercise 4 Let K be a cycle of length n with a chord (an edge between two non-neighbouring vertices on the cycle), and let $\chi : V(K) \rightarrow \{\text{red, blue}\}$ be a non-constant two-colouring that is *not* proper. Show that for every $\ell < n$ there exists a path of length ℓ in K whose endpoints are coloured differently.

[*Hint:* [to be read backwards] fl siht si ton eurt, redisnoc eht tsetrohs gnissim htgnel dna
wohs taht eht gniruoloc tsum eb cidoirep dnuora eht elcyc.]