Constructive Combinatorics So 2018
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Exercise Sheet 5

Due date: June 20th 12:30 PM

You should try to solve and write clear solutions to as many of the exercises as you can.

Exercise 1 Recall in Benson's construction we used the quadratic surface Q_4 in the projective space $PG(4,q)$ over the finite field $\mathbb{F}_q$, defined as $Q_4 = \{x : x_0^2 + x_1x_{-1} + x_2x_{-2} = 0\}$. Prove the following properties of Q_4.

(i) $|Q_4| = q^3 + q^2 + q + 1$.

(ii) Q_4 contains $q^3 + q^2 + q + 1$ lines.

(iii) Every point is contained in $q + 1$ lines of Q_4.

Exercise 2 Define a hypergraph with vertices $\mathcal{P}_* = \{(a,b,c) : a,b,c \in \mathbb{F}_q\}$, lines $\mathcal{L}_* = \{[d,e,f] : d,e,f \in \mathbb{F}_q\}$, such that $(a,b,c) \in [d,e,f]$ if and only if $e - b = da$ and $f - c = ea$. Prove that the incidence graph of this hypergraph is precisely Wenger's C_6-free construction.

Exercise 3 Let $q = 2^{2\alpha+1}$, and define π_* by

$$\pi_* : (a,b,c) \mapsto \left[a^{2^{\alpha+1}}, (ab)^{2^\alpha} + c^{2^\alpha}, b^{2^{\alpha+1}}\right],$$
$$\pi_* : [d,e,f] \mapsto \left(d^{2^\alpha}, f^{2^\alpha}, (df)^{2^\alpha} + e^{2^{\alpha+1}}\right).$$

Show that π_* is a polarity for the hypergraph defined in Exercise 2, and prove that the set of absolute points is

$$N_{\pi_*} := \left\{\left(a, b, a^{2^{\alpha+1}+2} + ab + b^{2^{\alpha+1}}\right) : a,b \in \mathbb{F}_q\right\}.$$

Finally, prove that the polarity graph Γ^{π_*} is C_6-free and hence $ex(n,C_6) \geq \frac{1}{2}n^{4/3} - \frac{1}{2}n^{2/3}$.

Exercise 4 Complete the doubling-trick proof of Füredi, Naor and Verstraëte to show $ex(n,C_6) \geq 0.534n^{4/3}$ for infinitely many n. Recall that the graph H was formed by taking a subset A of the vertices of the polarity graph Γ^{π_*} above, introducing twins of these vertices and twins of each edge incident to a vertex in A. Calculate the expected number of edges of H if A is chosen at random, with each vertex of Γ^{π_*} chosen independently with an appropriate probability p. Use this to deduce the claimed bound on $ex(n,C_6)$.