Constructive Combinatorics So 2018
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Exercise Sheet 6

Due date: July 4th

You should try to solve and write clear solutions to as many of the exercises as you can.

Exercise 1 Let $W_{4,q}$ be the four dimensional Wenger graph. Let $\theta(3,4)$ be the graph consisting of the union of three internally disjoint paths between two vertices.

- Characterize the copies of C_8 and $\theta(3,4)$ in $W_{4,q}$.
- Prove that $W_{4,q}$ is not vertex transitive.

Exercise 2 For $r, d \in \mathbb{N}$ let $T = T_{d,r}$ be the full d-ary tree of depth r, with root vertex w. In the lecture we defined a Cayley graph $C = C(S_V, S)$ on the symmetric group S_V of permutations of the vertex set $V = V(T)$. To specify the set $S = \{\pi_1, \ldots, \pi_d\}$ of generators, we fixed an arbitrary proper d-coloring $\chi : E(T) \to [d]$ and for each color $i = 1, \ldots, d$ we defined a permutation $\pi_i \in S_V$ as follows. For a vertex $u \in V(C)$ which has a neighbor z such that $\chi(uz) = i$, we set $\pi_i(u) = z$ (since χ is proper, there is no more than one such neighbor z). Otherwise, $\pi_i(u) = u$.

Prove that the girth of this graph is at least $4r + 2$

Exercise 3 Let G and H be two graphs, and let $G \otimes H$ be their Abbott product.

(i) Verify that $\alpha(G \otimes H) = \alpha(G) \cdot \alpha(H)$.

(ii) Prove that the Abbott product is associative.

We first constructed superpolynomially large Ramsey graphs by taking the Abbott product $\mathcal{G}_{n_0}$ of all labelled graphs on n_0 vertices.

(iii) Recalling the bounds we obtained on the number of vertices and the clique and independence numbers of $\mathcal{G}_n$, verify that this construction shows $R(k,k) \geq k^{\Omega\left(\frac{\log\log\log k}{\log\log\log\log k}\right)}$.

Exercise 4 The Paley graph P_q is defined on vertices $\mathbb{F}_q$, where q is a prime power congruent to 1 modulo 4, with $xy \in E$ if and only if $x - y$ is a quadratic residue in $\mathbb{F}_q$.

(i) Show that P_q is isomorphic to its complement.

(ii) Show that P_q is edge-transitive; that is, for every pair of edges xy and uv in E, there is an isomorphism of P_q mapping x to u and y to v.

(iii) Show that $R(4,4) \geq 18$.