## Erdős-Simonovits-Stone Theorem

**Theorem.** (Erdős-Stone, 1946) For arbitrary fixed integers $r \geq 2$ and $t \geq 1$

$$ex(n, T_{rt,r}) = \left(1 - \frac{1}{r-1}\right)\binom{n}{2} + o(n^2).$$

**Corollary.** (Erdős-Simonovits, 1966) For any graph $H$,

$$ex(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right)\binom{n}{2} + o(n^2).$$

**Corollaries of the Corollary.**

$$\begin{aligned} ex(n, \texttt{octahedron}) &= \frac{n^2}{4} + o(n^2) \\ ex(n, \texttt{dodecahedron}) &= \frac{n^2}{4} + o(n^2) \\ ex(n, \texttt{icosahedron}) &= \frac{n^2}{3} + o(n^2) \\ ex(n, \texttt{cube}) &= o(n^2) \end{aligned}$$

## Proof of the Erdős-Simonovits Corollary

**Theorem.** (Erdős-Stone, 1946) For arbitrary fixed integers $r \geq 2$ and $t \geq 1$

$$ex(n, T_{rt,r}) = \left(1 - \frac{1}{r-1}\right)\binom{n}{2} + o(n^2).$$

**Corollary.** (Erdős-Simonovits, 1966) For any graph $H$,

$$ex(n, H) = \left(1 - \frac{1}{\chi(H)-1}\right)\binom{n}{2} + o(n^2).$$

*Proof of the Corollary.* Let $r = \chi(H)$.

- $\chi(T_{n,r-1}) < \chi(H)$, so $e(T_{n,r-1}) \leq ex(n, H)$.
- $T_{r\alpha,r} \supseteq H$, so $ex(n, T_{r\alpha,r}) \geq ex(n, H)$, where $\alpha$ is a constant depending on $H$; say $\alpha = \alpha(H)$.

$\square$

## Proof of the Erdős-Stone Thm

**Erdős-Stone Theorem.** (Understanding precisely what it actually says) For any $\epsilon > 0$ and integers $r \geq 2$, $t \geq 1$ there exists an integer $M = M(r, t, \epsilon)$, such that any graph $G$ on $n \geq M$ vertices with more than $\left(1 - \frac{1}{r-1} + \epsilon\right)\binom{n}{2}$ edges contains $T_{rt,r}$.

We derive this through the following statement.

**Seemingly Weaker Theorem.** For any $\epsilon > 0$ and integers $r \geq 2$, $t \geq 1$ there exists an integer $N = N(r, t, \epsilon)$, such that any graph $G$ on $n \geq N$ vertices and with $\delta(G) \geq \left(1 - \frac{1}{r-1} + \epsilon\right) n$ contains $T_{rt,r}$.

Note that w.l.o.g. $\epsilon < \frac{1}{r-1}$.

*Derivation of the Erdős-Stone Theorem from the Seemingly Weaker Theorem.*

Let $G$ be a graph on $n \geq M(r,t,\epsilon)^*$ vertices with more than $\left(1-\frac{1}{r-1}+\epsilon\right)\binom{n}{2}$ edges. Recursively delete vertices which are adjacent to less than $\left(1-\frac{1}{r-1}+\frac{\epsilon}{2}\right)$-fraction of the remaining vertices.
What is the number $n'$ of vertices we are left with?

We deleted at most $\displaystyle\sum_{j=n'+1}^{n} j\left(1-\frac{1}{r-1}+\frac{\epsilon}{2}\right)$ edges. So

$$e(G) \leq \left(\binom{n+1}{2}-\binom{n'+1}{2}\right)\left(1-\frac{1}{r-1}+\frac{\epsilon}{2}\right)+\binom{n'}{2}.$$

This implies

$$\frac{\epsilon}{2}\binom{n}{2}-n \leq \left(\frac{1}{r-1}-\frac{\epsilon}{2}\right)\binom{n'}{2}-n'.$$

We choose $M(r,t,\epsilon)$ such that $n \geq M(r,t,\epsilon)$ implies $n' \geq N(r,t,\epsilon/2)$.

$^*$At this point we don't know $M(r,t,\epsilon)$ yet!!! We'll define it in the proof through $N(r,t,\epsilon/2)$. (which is known!)

*Proof of the Seemingly Weaker Theorem.*

Induction on $r$.

For $r = 2$ the claim is true provided $\frac{\binom{\epsilon n}{t} n}{\binom{n}{t}} > t - 1$, which is certainly true from some threshold $N(2, t, \epsilon)$.

Let $r \geq 2$ and $G$ be a graph on $n \geq N(r+1, t, \epsilon)^*$ vertices with $\delta(G) \geq \left(1 - \frac{1}{r} + \epsilon\right) n$.

We would like to find a $T_{(r+1)t, r+1}$ in $G$.

Let $s = \left\lceil \frac{t}{\epsilon} \right\rceil$. By the induction hypothesis$^\dagger$ there is a $T_{rs,r}$ in $G$ with vertex-set $A_1 \cup \ldots \cup A_r$, where $|A_1| = \ldots = |A_r| = s$.

$U = V(G) \setminus (A_1 \cup \ldots \cup A_r)$.

$W = \{w \in U : |N(w) \cap A_i| \geq t, i = 1, \ldots, r\}$ is the set of vertices eligible to extend some part of $A_1, \ldots, A_r$ into a $T_{(r+1)t, r+1}$.

$^*$Again, we don't know $N(r+1, t, \epsilon)$ yet.

$^\dagger$Here we assume $N(r+1, t, \epsilon) \geq N(r, s, \epsilon)$.

Double-count the number of edges missing between $U$ and $A_1 \cup \ldots \cup A_r$. They are

- at least $(|U| - |W|)(s-t)$ ($\approx (s-t)n$ if $W$ is small)
- at most $rs\left(\frac{1}{r} - \epsilon\right)n$ ($\approx (s-rt)n$, ↯ if $W$ is small)

From this we have

$$|W| \geq \frac{(r-1)\epsilon}{1-\epsilon}n - rs$$

Thus if $n$ is large enough* then

$$|W| > \binom{s}{t}^r (t-1).$$

So we can select $t$ vertices from $W$, which are adjacent to the same $t$ vertices in each $A_i$ .

*If $N(r+1, t, \epsilon) > \left(\binom{s}{t}^r (t-1) + rs\right) \frac{1-\epsilon}{(r-1)\epsilon}$