Finite Geometry So 2019
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Exercise Sheet 1

Due date: 14:15, 23rd April

You should try to solve all of the exercises below, but clearly mark which three solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1

(a) Let $\mathcal{S}$ be a linear space on n^2 points where each line is incident with exactly n points, for some integer $n \geq 2$. Prove that $\mathcal{S}$ is an affine plane.

(b) Prove that if a system of $n-1$ mutually orthogonal Latin squares of order n exists, then there exists an affine plane of order n.

Exercise 2

(a) Given two Latin squares of orders m and n, construct a Latin square of order mn.

(b) Given two orthogonal Latin squares of order m, and two orthogonal Latin squares of order n, construct two orthogonal Latin squares of order mn.

(c) Prove that for every $n \not\equiv 2 \pmod 4$, there exist two orthogonal Latin squares of order n.

(Hint: In part (a) try to construct the new Latin square over the ground set $S_1 \times S_2$, where S_1 is the ground set of the first Latin square and S_2 of the second Latin square.)

Exercise 3 For $n \geq 2$ an integer, define a graph G_n whose vertex set is equal to the set of all $n \times n$ Latin squares on the ground set $[n]$, and two Latin squares are adjacent if they are orthogonal to each other. Prove that for all n, the chromatic number $\chi(G_n)$ is at most $n-1$. For what values of n is the chromatic number equal to $n-1$?

Exercise 4 A subplane of a projective plane $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a projective plane $(\mathcal{P}', \mathcal{L}', \mathrm{I}')$ such that $\mathcal{P}' \subsetneq \mathcal{P}$, $\mathcal{L}' \subsetneq \mathcal{L}$ and $\mathrm{I}' = \mathrm{I} \cap (\mathcal{P}' \times \mathcal{L}')$.

(a) Prove that if a projective plane has order n and m is the order of a subplane of the projective plane, then $n \geq m^2$.

(b) Prove that the bound above is tight by constructing a projective plane of order m^2 and a subplane in it that has order m, for some m (preferably an infinite family of m's).

Exercise 5 For a prime power q, and integers $0 \leq k \leq n$, the Gaussian binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}.$$

(a) Prove that the number of k-dimensional subspaces of $\mathrm{PG}(n, q)$ is equal to $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q$.

(b) Give combinatorial proofs of the following identities:

(i)

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q.$$

(ii)

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q.$$

Exercise 6 For integers $1 \leq k \leq n$, prove the points of $\mathrm{PG}(n-1, q)$ can be partitioned into a collection of pairwise disjoint $k-1$ dimensional subspaces if and only if k divides n.

(Hint: There is a natural map from the vector space $\mathbb{F}_{q^k}^m$ to the vector space $\mathbb{F}_q^{km}$, where $km = n$, which uses the isomorphism between the additive structure of $\mathbb{F}_{q^k}$ and the vector space $\mathbb{F}_q^k$.)