Finite Geometry | So 2019

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Exercise Sheet 2

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You should try to solve all of the exercises below, but clearly mark which three solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1

(a) Prove that the Fano plane is not a subplane of $\mathrm{PG}(2, \mathbb{R})$.

(b) Give a characterisation of all fields F, in terms of the characteristic of F, for which the Fano plane is a subplane of $\mathrm{PG}(2, F)$.

Exercise 2 Let K_6 denote the complete graph on the vertex set $[6] := \{1, 2, 3, 4, 5, 6\}$. Define a point line geometry $(\mathcal{P}, \mathcal{L}, I)$ as follows:

- $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ where $\mathcal{P}_1 = [6]$ and $\mathcal{P}_2$ is the set of all distinct 1-factors (perfect matchings) of K_6.
- $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ where $\mathcal{L}_1 = \binom{[6]}{2}$ and $\mathcal{L}_2$ is the set of all distinct 1-factorizations of K_6.[1]
- a vertex $x \in \mathcal{P}_1$ is incident to an edge $e \in \mathcal{L}_1$ if $x \in e$ and it is never incident to an element of $\mathcal{L}_2$; a 1-factor $f \in \mathcal{P}_2$ is incident to an edge $e \in \mathcal{L}_1$ if $e \in f$ and it is incident to a 1-factorization $F \in \mathcal{L}_2$ if $f \in F$.

(a) Prove that this point-line geometry is a projective plane of order 4.

(b) Prove that this projective plane is isomorphic to $\mathrm{PG}(2, 4)$.

Exercise 3 In this exercise we give an alternate proof of the fact any oval in a finite projective plane of even order can be uniquely extended to a hyperoval. Let $\mathcal{O}$ be an oval in a finite projective plane of even order.

(a) Prove that through every point of the plane which lies on a secant of $\mathcal{O}$, i.e. a line intersecting $\mathcal{O}$ in exactly two points, there is a unique line which is tangent to $\mathcal{O}$.

(b) Deduce that all tangents of $\mathcal{O}$ intersect in a common point, and thus show that $\mathcal{O}$ can be uniquely extended to a hyperoval.

[1] A 1-factorization is a partition of the edges of K_6 into pairwise disjoint 1-factors.

Exercise 4 Let q be an odd prime power and let S be a set of points in $\mathrm{PG}(3, q)$ such that no three of them are collinear.

(a) Prove that $|S| \leq q^2 + 1$ with equality if and only if every plane of $\mathrm{PG}(3, q)$ intersects S in 0, 1 or $q + 1$ points.

(b) Prove that if $|S| = q^2 + 1$ then through every point x of S, there exists a unique plane π_x such that $S \cap \pi_x = \{x\}$ and moreover, for any line ℓ through x, we have $\ell \cap S = \{x\}$ if and only if $\ell \in \pi_x$.

(c) (Bonus) Construct such a set S with $|S| = q^2 + 1$, for every odd prime power q.

Exercise 5

(a) For an arbitrary field F, let S be a set of 5 points in $\mathrm{PG}(2, F)$ such that no three of them are collinear. Prove that there exists a unique conic containing S. Moreover, show that the conic is non-degenerate.

(b) Find the number of distinct non-degenerate conics in $\mathrm{PG}(2, q)$.

(c) For every even prime power $q \geq 8$, show that there exists an oval in $\mathrm{PG}(2, q)$ which is not a conic.

Exercise 6 Let F be a finite field of characteristic 2, and let C be a conic given by $ax^2 + by^2 + cz^2 + fyz + gzx + hxy = 0$.

(a) Prove that C is equal to a repeated line if and only if $f = g = h = 0$.

(b) Say C is not equal to a repeated line, and let N be the point of $\mathrm{PG}(2, F)$ with coordinates (f, g, h). Prove that C is singular if and only if N lies on C.

(c) Say C is irreducible. Then prove that every line tangent to C passes through the point N.

Hints:

Ex 1 Without loss of generality you can start by assuming that $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(1,1,1)$ are the coordinates of four points of the Fano subplane.[2] What are the coordinates of the other three points?

Ex 2 (b) A hyperoval in $\mathrm{PG}(2,4)$ has 6 points, and they exist.

Ex 4 Look at $S \cap \pi$ where π runs through the set of planes containing a well chosen line with respect to S.

Ex 5 (a) Look at the hint for Exercise 1. What can we assume about the coordinates of these five points?

[2] Prove this claim if you use it!