Finite Geometry So 2019

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Exercise Sheet 3

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You should try to solve all of the exercises below, but clearly mark which three solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 A blocking set with respect to $(n-1)$-dimensional subspaces in a projective or affine space of dimension n is a set of points such that every $(n-1)$-dimensional subspace meets the set non-trivially. Let $n \geq 2$ be an integer and q an arbitrary prime power.

(a) Find the minimum possible size of such a blocking set in $\mathrm{PG}(n, q)$, and classify the smallest examples.

(b) Find the minimum possible size of such a blocking set in $\mathrm{AG}(n, q)$.

Exercise 2 Let π be a projective plane of order q (not necessarily a prime power) and $\mathcal{L}$ be a subcollection of lines of π. A set B of points of π is called a blocking set relative to $\mathcal{L}$ if for every line ℓ in $\mathcal{L}$, we have $\ell \cap B \neq \emptyset$.

Let $\mathcal{O}$ be an oval of π and assume that q is odd. For each of the following subcollections $\mathcal{L}$ of lines, find the smallest possible size of a blocking set relative to $\mathcal{L}$.

(a) the set of all external lines to $\mathcal{O}$.

(b) the set of all secants and tangents to $\mathcal{O}$.

(Bonus) In each case characterise the smallest examples, assuming that $\pi \cong \mathrm{PG}(2, q)$.

Exercise 3 For a subset S of points in $\mathrm{AG}(2, q)$ we say that S determines a direction m if there exist two points in S such that the line joining them has slope m, where $m \in \mathbb{F}_q \cup \{\infty\}$ (vertical lines given by the equation $x = c$ are assumed to have slope ∞).

(a) Prove that if $|S| > q$, then S determines all directions.

(b) Let f be a function from $\mathbb{F}_q$ to $\mathbb{F}_q$. Consider the set $S = \{(x, f(x)) : x \in \mathbb{F}_q\}$ of points in $\mathrm{AG}(2, q)$ and let D be the set of directions determined by S. Prove that if $|D| > 1$, then the set $B_f = S \cup D$ is a non-trivial blocking set in the projective completion $\mathrm{PG}(2, q)$ of $\mathrm{AG}(2, q)$.

(c) Determine the size of the blocking set B_f when $f(x) = x^{(q+1)/2}$, assuming q to be odd.

Exercise 4 Let $f(q)$ be the smallest possible size of a set S of points in $\mathrm{AG}(3,q)$ such that every line intersects S non-trivially.

(a) Prove that

$$2q^2 - q \leq f(q) \leq 3q^2 - 3q + 1$$

for all q.

(b) Improve the lower bound to $2q^2 - 1$, for all q.

(Hint: enil nevig yna hguorht senalp $1 + q$ era ereht)

(Bonus) For q square improve the upper bound on $f(q)$ to $f(q) \leq 2q^2 + o(q^2)$, by using the fact that there exist 2 disjoint Baer subplanes in $\mathrm{PG}(2,q)$.

Exercise 5 Let $\mathcal{A}$ be an affine plane of order q^2, and let π be the projective plane of order q^2 obtained from $\mathcal{A}$ by adding a line ℓ_∞. Let D be a set of $q+1$ points on ℓ_∞ with the following property: for any two distinct points x, y of $\mathcal{A}$, if the line xy meets ℓ_∞ in a point of D, then there exists a Baer subplane B of π containing x and y such that $B \cap \ell_\infty = D$. Define the following point-line geometry $\mathcal{A}_D = (\mathcal{P}, \mathcal{L}, I)$:

(a) $\mathcal{P}$ is equal to the the point set of $\mathcal{A}$.

(b) $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$, where $\mathcal{L}_1$ is the set of lines of $\mathcal{A}$ that meet ℓ_∞ in a point outside D, and $\mathcal{L}_2$ is the set of Baer subplanes B of π that satisfy $B \cap \ell_\infty = D$.

(c) I is the natural incidence of containment.

Prove that $\mathcal{A}_D$ is an affine plane of order q^2. [1]

Exercise 6

(a) Let S be be a subset of points in a projective plane of order q, and let $k_1, \ldots, k_{q^2+q+1}$ be the intersection sizes of the q^2+q+1 lines with the set S. Prove that

$$\sum_{i=1}^{q^2+q+1} \left(k_i - \frac{(q+1)|S|}{q^2+q+1}\right)^2 \leq q|S|.$$

(b) Let S be a set of points and T a set of lines in a projective plane of order q. Define $i(S,T)$ to be the number of incidences between the points of S and the lines of T, that is, $i(S,T) = |I \cap (S \times T)|$, where I is the point-line incidence relation of the projective plane. Prove that

$$\left| i(S,T) - \frac{q+1}{q^2+q+1}|S||T| \right| \leq \sqrt{q|S||T|}.$$

[1] By choosing such sets D in $\mathrm{PG}(2,q^2)$ one can construct non-Desarguesian planes of order q^2. You can try to find such a set and show that the plane you construct is non-Desarguesian, for all $q \geq 3$.