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Exercise Sheet 4

Due date: 14:15, 4th June

You should try to solve all of the exercises below, but clearly mark which three solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Let $\mathcal{H} = \{(x, y, z) \in \mathrm{PG}(2, q^2) : x^{q+1} + y^{q+1} + z^{q+1} = 0\}$.

(a) Determine $|\mathcal{H}|$ by considering the following three types of points in the plane that can lie on $\mathcal{H}$: $\{(x, 1, 0) : x \in \mathbb{F}_{q^2}\}$, $\{(0, y, 1) : y \in \mathbb{F}_{q^2}\}$ and $\{(x, y, 1) : x, y \in \mathbb{F}_{q^2}, x \neq 0\}$.

[Hint (to be read backwards): .$\mathbb{F}_q$ dleifbus sti ot $\mathbb{F}_{q^2}$ morf noitcnuf mron eht ta kooL]

For the next two parts assume that $\mathcal{H}$ does not contain any line completely.

(b) For an arbitrary point P of $\mathcal{H}$ find a line ℓ_P through P that is tangent to $\mathcal{H}$.

(c) Show that every line meets $\mathcal{H}$ in exactly 1 or $q + 1$ points and determine the number of lines of each kind through every point of $\mathrm{PG}(2, q^2)$.

(Bonus) Show that $\mathcal{H}$ does not contain any line completely.

Exercise 2 Prove that the following graphs are strongly regular and determine their parameters.

(a) Given a set of $m - 2$ MOLS of order n, $m \geq 2$, the graph $L_m(n)$ with vertex set $[n]^2$ and two vertices adjacent if they either share a coordinate, or have the same symbol appearing in the coordinates in one of the Latin squares (if $m > 2$).

(b) The graph on the lines of $\mathrm{PG}(n, q)$, $n \geq 3$, with two lines adjacent if they meet in a point.

(c) Let $P_1, \ldots, P_5, Q_1, \ldots, Q_5$ be disjoint copies of the cycle graph C_5. Label the vertices of P_i as $\{v_{i1}, v_{i2}, v_{i3}, v_{i4}, v_{i5}\}$ such that v_{ik} is adjacent to $v_{i(k+1)}$. Label the vertices of Q_j as $\{u_{j1}, u_{j2}, u_{j3}, u_{j4}, u_{j5}\}$ such that u_{jk} is adjacent to $u_{j(k+2)}$.[1] For all $1 \leq i, j, k \leq 5$ connect v_{ik} to $u_{jk'}$ with an edge where $k' = ij + k$ (the arithmetic on all indices is done modulo 5).

[1] With this labelling P_i's look like pentagons whereas Q_j's look like pentagrams.

Exercise 3 Let G be an $\mathrm{srg}(n, k, \lambda, \mu)$.

(a) Prove that $\mu \leq k$, with equality if and only if G is a complete multipartite graph.

(b) Prove that the following are equivalent: (i) $\mu = 0$, (ii) $k = \lambda + 1$, (iii) G is a disjoint union of complete graphs, (iv) G is disconnected.

Exercise 4 Let G be a graph with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

(a) Prove that

$$\lambda_1 = \max_{||x||=1} x^T Ax = \max_{x \neq 0} \frac{x^T Ax}{||x||^2},$$

and deduce that $\lambda_1 \geq \overline{d}(G)$, that is, the average degree of G.

(b) Suppose G is k-regular and connected. Prove that G is bipartite if and only if $\lambda_n = -k$.

Exercise 5 Let G be a graph with $[n]$ as its vertex set, and let A be its adjacency matrix. Let $C = [2m]$ for some $1 \leq m \leq n/2$, and $D = [n] \setminus C$, such that the following two properties hold: (i) the induced subgraph on the vertex set C is regular and (ii) every vertex in D is adjacent to 0, m or $2m$ vertices in C. Let

$$Q = \begin{pmatrix} \frac{1}{m} J_{2m} - I_{2m} & 0 \\ 0 & I_{n-2m} \end{pmatrix}$$

(a) Prove that QAQ^T is an adjacency matrix of a simple graph G' such that $\mathrm{Spec}(G') = \mathrm{Spec(G)}$.

(b) Use this idea to construct a pair of non-isomorphic graphs on 9 vertices that have the same spectrum.

Exercise 6 Let q be an odd prime power, and define a directed graph on $\mathbb{F}_q$ by taking (a, b) as an edge if $a - b$ is a square in $\mathbb{F}_q$.

(a) Prove that this graph is an undirected graph if and only if $q \equiv 1 \pmod 4$.

(b) Prove that for $q \equiv 1 \pmod 4$, this graph is a strongly regular graph and compute its parameters.

(c) Find the independence number of the graph when $q \equiv 1 \pmod 4$ and q is an even power of a prime.

Exercise 7 [Bonus]

(a) Let $\mathcal{O}$ be an oval in a finite projective plane of order 5. Prove that the graph defined on the interior points of the plane, that is, the points through which there are no tangents to $\mathcal{O}$, with two points adjacent if the line joining them is a secant of $\mathcal{O}$, is isomorphic to the Petersen Graph.

(b) Use this to identify all points and lines of the projective plane with certain substructures of the Petersen graph (vertices, edges, 1-factors, etc.), and thus prove the uniqueness of the projective plane of order 5 assuming it contains an oval.

(c) Show that every projective plane of order 5 has an oval.