Finite Geometry So 2019
Anurag Bishnoi
Tibor Szabó

Exercise Sheet 5

Due date: 14:15, 18th June

You should try to solve all of the exercises below, but clearly mark which three solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Let S be a two-intersection set in $\mathrm{PG}(2,q)$, with every line meeting S in either k_1 or k_2 points, $k_1 < k_2$ (and both values occur).

(a) Show that $k_2 - k_1$ divides q.

(b) Prove that $|S|$ is one of the roots of the quadratic equation

$$x^2 - (q(k_1+k_2-1)+k_1+k_2)x + k_1k_2(q^2+q+1) = 0.$$

Exercise 2 Let σ be the polarity of $\mathrm{PG}(2,q)$ given by $(a,b,c) \mapsto \{(x,y,z) : ax+by+cz = 0\}$, that has $q+1$ absolute points, i.e., points x that are contained in their image $\sigma(x)$. Let G be the graph defined on the points of $\mathrm{PG}(2,q)$ with $x \sim y$ if $x \in \sigma(y)$, and let G' be the graph obtained from G by removing the $q+1$ loops at the absolute points.

(a) Show that G' contains no C_4's as subgraphs, and determine the number of edges in G'.[1]

(b) Prove that there are no triangles in G' containing an absolute point, and for each non-absolute point determine the number of triangles it is contained in.

Exercise 3 An ovoid in $\mathrm{PG}(3,q)$ is a set of q^2+1 points, no three of which are collinear. Let O be an ovoid in a hyperplane of $\mathrm{PG}(4,q)$. Give a construction of a generalized quadrangle of order (q,q^2) using O, and prove the correctness of your construction.

Exercise 4 Let $\beta : \mathbb{F}_{q^2}^4 \times \mathbb{F}_{q^2}^4 \to \mathbb{F}_{q^2}^4$ be defined as follows,

$$\beta((x_0,x_1,x_2,x_3),(y_0,y_1,y_2,y_3)) = x_0y_0^q + x_1y_1^q + x_2y_2^q + x_3y_3^q.$$

A subspace S of $\mathrm{PG}(3,q^2)$ is called totally isotropic with respect to β if $\beta(x,y) = 0$ for all $x,y \in S$. Prove that the point-line geometry of the totally isotropic points, and the totally isotropic lines in $\mathrm{PG}(3,q)$ is a generalized quadrangle. Also determine the order of this generalized quadrangle.

[1] This graph, originally studied by Erdős and Rényi, in fact gives the densest possible graph on q^2+q+1 vertices without a C_4, for every large enough prime power q. It can be also be used to construct the asymptotically densest possible C_4-free graphs on n vertices, for any large enough integer n.

Exercise 5 Let G be a non-empty finite simple graph with the following property:

Every edge xy of G is contained in a triangle x, y, z with the property that any vertex $u \notin \{x, y, z\}$ is adjacent to exactly one of x, y, z.

(a) Prove that G is either the Windmill graph, or a k-regular graph with $k \in \{4, 6, 10\}$.

(b) For each of these values of k, construct such a graph G.

(Bonus) Prove (a) without assuming that the graph is finite.

Exercise 6 Borsuk's conjecture[2] states that any bounded subset S of $\mathbb{R}^d$, consisting of at least 2 points, can be partitioned into $d+1$ subsets of smaller diameter, for all $d \geq 2$.[3]

(a) Let G be a strongly regular graph on n vertices with

$$\mathrm{Spec}(G) = \begin{pmatrix} k & \theta_1 & \theta_2 \\ 1 & m_1 & m_2 \end{pmatrix}.$$

Construct a finite set S of n points in $\mathbb{R}^{m_1}$ that cannot be partitioned into less than $n/\omega(G)$ parts of smaller diameter, where $\omega(G)$ is the clique number of G.

(b) Use the existence of an srg$(416, 100, 36, 20)$ with clique number equal to 5 to disprove Borsuk's conjecture.[4] [5] [6]

[2] now known to be false in general

[3] You should try to see why d such subsets would clearly not suffice.

[4] This strongly regular graph is related to many interesting finite simple groups and finite geometries, but it's difficult to give a well motivated construction here.

[5] You'll see another way of disproving Borsuk's conjecture in the course Extremal Combinatorics next semester.

[6] Since the number 6 has played an important role in our course, I thought I'll add this extra footnote just so that we reach the 6'th footnote in this heavily footnoted exercise.