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Exercise Sheet 6

Due date: 14:15, 2nd July

You should try to solve all of the exercises below, but clearly mark which three solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Consider a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ for which the following conditions hold, with $s, t, \alpha \geq 1$ as constants§:

- $\mathcal{S}$ has order (s, t).
- For any $(x, \ell) \notin I$, x is collinear with exactly α points of ℓ.

(a) Prove that if $\alpha \neq s+1$, then the collinearity graph of $\mathcal{S}$ is strongly regular and determine its parameters.

(b) Let K be a set of points in $\mathrm{PG}(2, q)$ for which every line ℓ intersects K in 0 or d points, for some $d \in \{2, 3, \ldots, q-1\}$. Define the point-line geometry $\mathcal{S}(K)$ with points as the set of points of $\mathrm{PG}(2, q)$ outside K, and lines as those lines that intersect K in exactly d points (natural incidence). Prove that $\mathcal{S}(K)$ is a point-line geometry satisfying the conditions above, and determine the parameters s, t, α of $\mathcal{S}(K)$. When is $\mathcal{S}(K)$ a generalized quadrangle?

Exercise 2 A connected graph G of diameter d is called distance regular if there exist constants a_i, b_i, c_i, with $i \in \{0, 1, \ldots, d\}$ such that the following holds for any two vertices x and y of G at distance i from each other:

- there are a_i neighbours of y at distance i from x;
- there are b_i neighbours of y at distance $i+1$ from x;
- there are c_i neighbours of y at distance $i-1$ from x.

Clearly $a_0 = c_0 = b_d = 0$ and G is k-regular with $k = a_0 + b_0 + c_0 = \cdots = a_d + b_d + c_d$.

(a) Prove that the collinearity graph of a generalized n-gon of order (s, t) is a distance regular graph of diameter $\lfloor \frac{n}{2} \rfloor$.

(b) Determine the number of points and lines in a generalized n-gon of order (s, t).

Exercise 3

(a) Prove that a k-regular graph of diameter d has at most

$$1 + k\sum_{i=0}^{d-1}(k-1)^i$$

vertices, with equality if and only if the graph is a Moore graph.

(b) Prove that a k-regular bipartite graph of diameter d has at most

$$2\sum_{i=0}^{d-1}(k-1)^i$$

vertices, with equality if and only if the graph is the incidence graph of a generalized d-gon of order $(k-1, k-1)$.

Exercise 4 A *near* $2d$-*gon*, for $d \geq 2$, is a partial linear space whose collinearity graph has diameter d, and for every point x and every line ℓ, there exists a unique point $x' \in \ell$ that is closest to x in the collinearity graph, among all points of ℓ.

(a) Prove that every generalized $2d$-gon is a near $2d$-gon.

(b) Prove that every near $2d$-gon that satisfies the following two properties is a generalized $2d$-gon: (i) every point is incident with at least two lines, and (ii) for every pair of points x and y at distance $2 \leq i \leq d-1$ from each other in the collinearity graph there is a unique neighbour of y at distance $i-1$ from x.

Exercise 5 Let G be the incidence graph of a non-thick generalized d-gon. Assume that G is not a cycle of length $2d$. Let k be the minimum distance between two thick vertices of G.

(a) Prove that if $k = d$, then G is the k-fold subdivision of a graph consisting of 2 vertices and at least 3 edges between them, that is, a multiple edge of multiplicity at least 3.

(b) If $k < d$, show that G is the k-fold subdivision of the incidence graph G' of a thick generalized d'-gon, with $d' = d/k$.

[Hint (to be read backwards): .G' ni $2d'$ htgnel fo elcyc a fo noisividbus dlof-k a si G ni $2d$ htgnel fo elcyc yreve taht wohS]