Finite Geometry So 2019
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Exercise Sheet 7

Due date: 14:15, 9th July

You should try to solve all of the exercises below, but clearly mark which three solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 An ovoid $\mathcal{O}$ in $\mathrm{PG}(n, q)$ is a set of points with the following two properties: (i) no three distinct points of $\mathcal{O}$ are collinear, and (ii) for all points $x \in \mathcal{O}$ there exists a hyperplane H_x such that the set of lines through x that are tangent to $\mathcal{O}$ is equal to the set of lines through x in H_x.

(a) Prove that an ovoid in $\mathrm{PG}(n, q)$ has $q^{n-1} + 1$ points.

(b) Prove that for every $n \geq 4$, $\mathrm{PG}(n, q)$ does not contain any ovoids.

Exercise 2

(a) Prove that a bipartite k-regular graph of girth $g \geq 4$ has at least

$$2 \sum_{i=0}^{(g-2)/2} (k-1)^i,$$

vertices, and for every $g \in \{4, 6, 8, 12\}$ the bound is sharp for infinitely many values of k.

(b) For every prime power q, construct a bipartite q-regular graph of girth 6 with $2q^2$ vertices, and of girth 8 with $2q^3$ vertices.

(c) (Bonus) Improve the constructions from (b) to get as close as you can to the lower bounds of $2(q^2 - q + 1)$ and $2(q^3 - 2q^2 + 2q)$, respectively.

Exercise 3 Let π be the Fano plane PG$(2,2)$. Construct the following point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L})$.

- $\mathcal{P}$ is equal to the set of all points, lines, and all point-line pairs (p, ℓ) of π.
- $\mathcal{L}$ is equal to the set of all the following 3-element subsets of $\mathcal{P}$: for every incident point-line pair (p, ℓ) in π, the sets $\{p, \ell, (p, \ell)\}$, $\{(p, \ell), (p_1, \ell_1), (p_2, \ell_2)\}$ and $\{(p, \ell), (p_1, \ell_2), (p_2, \ell_1)\}$, where p, p_1, p_2 are the points on ℓ in π and ℓ, ℓ_1, ℓ_2 are the lines through p in π.

(a) Determine the number of points and the number of lines in $\mathcal{S}$ and prove that it is a partial-linear space of order $(2, 2)$.

(b) Prove that $\mathcal{S}$ is a generalized hexagon.

Exercise 4 A bilinear form $\beta : V \times V \to F$, for some vector space V over F, is called *alternating* if $\beta(u, u) = 0$ for all $u \in V$. It is called non-degenerate if $\beta(u, v) = 0$ for all $v \in V$ implies that $u = 0$.

(a) Show that if β is an alternating bilinear form then $\beta(u, v) = -\beta(v, u)$ for all u, v.

(b) Prove that the maximum vector space dimension of a totally isotropic subspace with respect to a non-degenerate alternating bilinear form over a vector space of dimension $n + 1$ is equal to $(n + 1)/2$, and hence deduce that n must be odd.

(c) Prove that the totally isotropic points and lines of PG(n, F) with respect to a non-degenerate alternating bilinear form over the underlying vector space F^{n+1}, form a non-degenerate polar space. Also determine the order (s, t) of this polar space if $F = \mathbb{F}_q$.

Exercise 5 Let $\mathcal{S}$ be an orthogonal polar space of rank r and type ϵ, over $\mathbb{F}_q$. Prove that the number of totally singular subspaces of vector space dimension r is equal to

$$\prod_{i=1}^{r} (q^{i+\epsilon} + 1).$$

Exercise 6

(a) Give an example of a non-degenerate symmetric bilinear form that contains totally isotropic points and lines, such that these point and lines give rise to a degenerate polar space.

(b) Prove that a quadratic form over a field of characteristic not equal to 2 is non-degenerate if and only if the bilinear form associated with it is non-degenerate.

(c) Give an example of a non-degenerate quadratic form Q such that the bilinear form β associated with it is degenerate.

Exercise 7 (Bonus 10 points) Let β be a non-degenerate symmetric bilinear form over $\mathbb{F}_q^5$, that gives a polarity $\perp$ of $\mathrm{PG}(4,q)$ by mapping a point x to the hyperplane $x^\perp = \{y \in \mathrm{PG}(4,q) : \beta(x,y) = 0\}$. Let z be a point of $\mathrm{PG}(4,q)$ such that $z \notin z^\perp$, and let $\mathcal{O}$ be an ovoid in $z^\perp \cong \mathrm{PG}(3,q)$.

Define a graph G with vertex set equal to the set of points x in $\mathrm{PG}(4,q) \setminus (z^\perp \cup \{z\})$ such that x lies on a line joining z and a point of $\mathcal{O}$, and making two vertices x, y adjacent if $x \in y^\perp$.

(a) Determine the number of vertices n in G and show that the number of edges is at least $Cn^{5/3}$, for some constant C and large enough n.

(b) Prove that G does not contain any copies of the graph $K_{3,3}$.[1]

[1] In fact, the graph G is asymptotically the densest possible graph on n vertices that does not contain a $K_{3,3}$.