The Probabilistic Method | So 2020
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Exercise Sheet 2

Due date: 10:30, May 14th, to be submitted in Whiteboard.
Late submissions, like a comedy club's patrons on open mic night, will not be entertained.

You should try to solve all of the exercises below, and clearly mark which two solutions you would like to be graded — each problem is worth 10 points. Starred exercises represent problems that may be a little tougher, should you wish to challenge yourself.

Exercise 1 Using the method of alterations, derive an asymptotic lower bound for the off-diagonal Ramsey number $R(\ell, k)$ when $\ell \in \mathbb{N}$ is fixed and $k \to \infty$.

Exercise 2 One can more generally define the Ramsey number of a graph H, denoted $R(H)$, as the minimum n such that any red/blue colouring of the edges of K_n contains a monochromatic copy of H.

Let Q_d be the d-dimensional hypercube on the vertex set $V(Q_d) = \{0,1\}^d$, with an edge between two vertices if they differ in precisely one coordinate. Prove that $R(Q_d) \leq 2^{3d}$. [1]

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2020/hints/S2.html`.]

Exercise 3

(a) Give an example where Chebyshev's Inequality is tight.

(b) Show that our corollary of Chebyshev's Inequality can be strengthened: if X is a nonnegative random variable, prove that

$$\mathbb{P}(X = 0) \leq \frac{\mathrm{Var}(X)}{\mathbb{E}[X^2]}.$$

[1] It is an open problem of Burr and Erdős to show that $R(Q_d) = O(2^d)$. The current best-known bound is $R(Q_d) \leq O(d2^{2d})$; bonus points for improving the constant in the exponent further.

Exercise 4 Given a graph G, the *domination number* $\gamma(G)$ is the size of the smallest dominating set in $V(G)$. Here we consider the domination number $\gamma(G(n,p))$ of the Erdős–Rényi random graph.

(a) Prove that when $\omega(n^{-1}) = p = o(1)$, we have $\gamma(G(n,p)) = \Omega\left(p^{-1}\log(np)\right)$ with high probability.[2]

(b) Deduce that our upper bound on the domination number of n-vertex graphs with minimum degree δ is of the correct order of magnitude when $\delta \to \infty$.

In case it simplifies your calculations, you may assume in both parts that $p = \omega(n^{-1/2})$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2020/hints/S2.html`.]

Exercise 5 Given a 3-uniform hypergraph $H = (V, E)$, the *matching number* $\nu(H)$ is the maximum number of pairwise-disjoint edges in $E(H)$. The *cover number* $\tau(H)$ is the size of the smallest set of vertices $A \subseteq V(H)$ such that for every edge $e \in E(H)$ we have $e \cap A \neq \emptyset$.

(a) Prove that for all 3-uniform hypergraphs H, $\tau(H) \leq 3\nu(H)$.

A *sail* S is a 3-uniform hypergraph with four edges arranged as follows: there is a vertex $w \in V(S)$ contained in three of the edges, $e_1, e_2, e_3 \in E(S)$. These three edges contain unique vertices $x \in e_1 \setminus (e_2 \cup e_3)$, $y \in e_2 \setminus (e_1 \cup e_3)$, and $z \in e_3 \setminus (e_1 \cup e_2)$. Finally, the fourth edge $e_4 = \{x, y, z\}$ contains precisely these three vertices. Note that there are, up to isomorphism, three different types of sails, depending on which of the edges e_1, e_2 and e_3 share their third vertices.

Aharoni and Zerbib asked whether, given a 3-uniform hypergraph H that does not contain a sail as a subhypergraph, we in fact have the stronger inequality $\tau(H) \leq 2\nu(H)$. [3]

(b) Answer their question negatively by showing that, for every $\varepsilon > 0$ and n sufficiently large, there is a n-vertex sail-free 3-uniform hypergraph with $\tau(H) \geq (3 - \varepsilon)\nu(H)$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2020/hints/S2.html`.]

[2] Here, and elsewhere, "with high probability" means with probability $1 - o(1)$.

[3] It is reasonable to ask if this is a reasonable thing to ask; at first sight, the existence or nonexistence of the sail has nothing to do with matchings or covers. However, often in graph theory one finds interesting classes of graphs are characterised by a set of forbidden subgraphs.[4]

There has been considerable interest in 3-uniform hypergraphs for which $\tau(H) \leq 2\nu(H)$. As part of a more general conjecture, Ryser postulated that this should hold for any 3-partite 3-uniform hypergraph — that is, one where the vertices can be partitioned into three classes, with each edge having precisely one vertex from each class. This was later proven by Aharoni using topological methods; a proof of the general conjecture (regarding r-partite r-uniform hypergraphs) eludes us.

Another famous conjecture in this direction is due to Tuza. Given a graph G, form a 3-uniform hypergraph H whose vertices are the edges of G, and whose edges are the triangles of G. Tuza conjectured that $\tau(H) \leq 2\nu(H)$ should hold for any such hypergraph, and this conjecture remains open.

It is not hard to see that neither the 3-partite hypergraphs of Ryser nor the triangular hypergraphs of Tuza can contain a sail as a subhypergraph. Hence, had Aharoni and Zerbib's question had a positive answer, it would have provided a simultaneous resolution of the two conjectures.

[4] For instance, planar graphs are precisely those graphs without $K_{3,3}$ or K_5 minors.

Exercise 6* Prove that there is an absolute constant $c > 0$ such that if $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$ satisfy $\sum_{i=1}^n a_i^2 = 1$, then for a uniformly random vector $(\xi_1, \xi_2, \ldots, \xi_n) \in \{-1, 1\}^n$, we have

$$\mathbb{P}\left(\left|\sum_{i=1}^n \xi_i a_i\right| \leq 1\right) \geq c.$$