Exercise Sheet 3

Due date: 10:30, May 28th, to be submitted in Whiteboard.
Late submissions will, like this year's travel plans, be cruelly discarded.

You should try to solve all of the exercises below, and clearly mark which two solutions you would like to be graded — each problem is worth 10 points. Starred exercises represent problems that may be a little tougher, should you wish to challenge yourself.

Exercise 1 Prove that there is some function $k : \mathbb{N} \to \mathbb{N}$ such that, as $n \to \infty$, we have $\omega(G(n, \frac{1}{2})) \in \{k(n) - 1, k(n), k(n) + 1\}$ with high probability, where $\omega(G(n, \frac{1}{2}))$ is the clique number of $G(n, \frac{1}{2})$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2020/hints/S3.html`.]

Exercise 2 Determine the threshold for $G(n, p)$ to be connected.

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2020/hints/S3.html`.]

Exercise 3 Suppose we have a collection of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in [n]^2$ such that there are no disjoint non-empty sets $I, J \subseteq [m]$ with $\sum_{i \in I} \vec{v}_i = \sum_{j \in J} \vec{v}_j$.

(a) Prove that we must have $m \leq 2 \log n + \log \log n + O(1)$.

(b) Show that we can have $m \geq 2 \lfloor \log n \rfloor + 1$.

Exercise 4 Let G be a graph, and suppose each vertex $v \in V(G)$ is assigned a list $L(v)$ of at least $10d$ colours, for some $d \geq 1$. Show that if, for every vertex v and $c \in L(v)$, there are at most d neighbours u of v such that $c \in L(u)$ as well, then there is a proper colouring φ of the vertices of G such that $\varphi(v) \in L(v)$ for each vertex v.

Exercise 5 Given sufficiently large $k \in \mathbb{N}$, let S be a finite subset of $\mathbb{R}$ with $|S| \geq 4k \ln k$. Prove that there is some k-colouring of $\mathbb{R}$ such that for every $x \in \mathbb{R}$, all k colours appear in the translation $x + S$ of the set S.

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2020/hints/S3.html`.]

Exercise 6* Let $f(k)$ denote the maximum s such that any k-SAT formula in which each variable appears in at most s clauses is satisfiable.

(a) Improve the bound from lecture to show that $f(k) \geq \left\lfloor \frac{2^{k+1}}{(k+1)e} \right\rfloor$.

(b) Construct an unsatisfiable k-SAT that shows $f(k) = O\left(\frac{2^k \log^2 k}{k}\right)$.