Exercise Sheet 4

Due date: 10:30, Jun 11th, to be submitted in Whiteboard.
Late submissions will be ignored more than my dietitian's advice.

You should try to solve all of the exercises below, and clearly mark which two solutions you would like to be graded — each problem is worth 10 points. Starred exercises represent problems that may be a little tougher, should you wish to challenge yourself. In case you have difficulties submitting in Whiteboard, please send your solutions to probmethod@gmail.com.

Exercise 1 In this exercise, we prove the bound on the lower tail probability from the asymmetric Chernoff bound (the upper tail is very similar, but with some additional technical considerations). Recall that we have values $p_1, p_2, \dots, p_n \in [0,1]$, and independent random variables X_i such that

$$X_i = \begin{cases} 1 - p_i & \text{with probability } p_i, \\ -p_i & \text{with probability } 1 - p_i, \end{cases}$$

and we set $X = \sum_{i=1}^n X_i$ and $p = \frac{1}{n}\sum_{i=1}^n p_i$. Given some $a > 0$, our goal is to bound $\mathbb{P}(X \le -a)$.

(a) Show that $\mathbb{E}[e^{-\lambda X}] \le e^{\lambda pn}(pe^{-\lambda}+(1-p))^n$, and deduce $\mathbb{P}(X \le -a) \le e^{\lambda pn + np(e^{-\lambda}-1)-\lambda a}$.

(b) Using the inequality $e^{-\lambda} \le 1-\lambda+\lambda^2/2$ for $\lambda \ge 0$, conclude that $\mathbb{P}(X \le -a) \le e^{-a^2/2pn}$.

Exercise 2 Run the edge-alteration argument to obtain a lower bound for the Ramsey number $R(4,k)$. How does this compare to the lower bound the Lovász Local Lemma gives?

Exercise 3 Although it is well-known that graphs with large chromatic number need not have large cliques, Hajós conjectured that they must nevertheless have a large hidden clique, in some sense. To be more precise, the Hajós number $h(G)$ of a graph G is the largest k for which G contains a K_k subdivision, where a subdivision of a graph H is obtained by replacing edges of H with internally vertex-disjoint paths of arbitrary length.

Hajós conjectured that for every graph G, we have $\chi(G) \le h(G)$. Refute his conjecture spectacularly by showing that there are graphs G for which the ratio $\chi(G)/h(G)$ is arbitrarily large.

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2020/hints/S4.html`.]

Exercise 4 Given some $\ell \in \mathbb{N}$ and $\varepsilon > 0$, an infinite sequence $(a_1, a_2, a_3, \ldots) \in \{0,1\}^{\mathbb{N}}$ of bits is called (ℓ, ε)-interesting if, for each $i \in \mathbb{N}$, the two subsequences $(a_i, a_{i+1}, \ldots, a_{i+\ell-1})$ and $(a_{i+\ell}, a_{i+\ell+1}, \ldots, a_{i+2\ell-1})$ differ in at least $(\frac{1}{2} - \varepsilon)\ell$ coordinates.

Prove that for every $\varepsilon > 0$ there is some $\ell_0 = \ell_0(\varepsilon)$ and a sequence $(a_1, a_2, a_3, \ldots) \in \{0,1\}^{\mathbb{N}}$ that is (ℓ, ε)-interesting for every $\ell \geq \ell_0$.

Exercise 5 Given a map $\chi : [n] \to \{-1, 1\}$, we write $\chi(F) = \sum_{x \in F} \chi(x)$ for a subset $F \subseteq [n]$. Given a family $\mathcal{F} \subseteq 2^{[n]}$ of subsets, we write $\chi(\mathcal{F}) = \max_{F \in \mathcal{F}} |\chi(F)|$. The discrepancy of $\mathcal{F}$ is defined as $\operatorname{disc}(\mathcal{F}) = \min_{\chi} \chi(\mathcal{F})$.

(a) Prove that if $|\mathcal{F}| = m$, we have $\operatorname{disc}(\mathcal{F}) \leq \sqrt{2n \ln(2m)}$.

(b) Show that if every $x \in [n]$ is contained in at most d members of $\mathcal{F}$, then we have $\operatorname{disc}(\mathcal{F}) \leq 2d - 1$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2020/hints/S4.html`.]

Exercise 6* [Warning: This exercise contains spoilers for Exercise 6 from the previous sheet. If you do not want any assistance in solving that problem, do not read this exercise.]

Let $s = \left\lfloor \frac{2^{k+1}}{e(k+1)} \right\rfloor$, and suppose we have a k-SAT formula in the variables $x_1, x_2, \ldots, x_t$ in which each variable appears in at most s clauses. For each variable x_i, let n_i denote the number of times the negated literal $\neg x_i$ appears in the formula.

By setting the variable x_i to be true with probability $p_i := \frac{1}{2} + \frac{2n_i - s}{2sk}$, use the Lopsided Lovász Local Lemma to show that the formula is satisfied with positive probability.