Exercise Sheet 5

Due date: 10:30, Jun 25th, to be submitted in Whiteboard.
Late submissions will receive as much credit as Rosalind Franklin did.

You should try to solve all of the exercises below, and clearly mark which two solutions you would like to be graded — each problem is worth 10 points. Starred exercises represent problems that may be a little tougher, should you wish to challenge yourself. In case you have difficulties submitting in Whiteboard, please send your solutions to probmethod@gmail.com.

Exercise 1 Depth-first search is a graph exploration algorithm that uncovers the connected components of a graph. Given a graph $G = (V, E)$, we build a sequence of partitions $V = U_i \cup A_i \cup E_i$ of the vertex set V into a set U_i of unexplored vertices, a set A_i of active vertices, and a set E_i of explored vertices. We initially set $U_0 := V$ and $A_0, E_0 := \emptyset$.

For each $i \geq 0$, the ith round of the algorithm proceeds as follows:

1. If $A_i = U_i = \emptyset$, terminate the algorithm — all vertices have been explored.
2. Otherwise, if $A_i = \emptyset$, let $u \in U_i$ be an arbitrary vertex. Make u active by setting $U_{i+1} := U_i \setminus \{u\}$, $A_{i+1} := \{u\}$, and $E_{i+1} := E_i$. Proceed to the next round.
3. Otherwise, if $A_i \neq \emptyset$, let $a \in A_i$ be the vertex most recently made active.
   3.1. If a has unexplored neighbours, let $u \in U_i$ be an arbitrary neighbour of a in U_i. We make u active by setting $U_{i+1} := U_i \setminus \{u\}$, $A_{i+1} := A_i \cup \{u\}$, and $E_{i+1} := E_i$, and proceed to the next round.
   3.2. Otherwise, if a has no unexplored neighbours, we mark a as inactive and move it to the explored set by taking $E_{i+1} := E_i \cup \{a\}$, $A_{i+1} := A_i \setminus \{a\}$ and $U_{i+1} := U_i$. Proceed to the next round.

For the final bit of set-up for this problem, we define a graph G to be (k, k)-*neighbourly* if, for any two disjoint sets A and B of vertices, each of size k, there is an edge from A to B.

(a) Show that every (k, k)-neighbourly graph on n vertices contains a path on $n - 2k + 1$ vertices and a cycle of length at least $n - 4k + 3$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2020/hints/S5.html`.]

(b) Deduce that $G\left(n, \omega\left(\frac{1}{n}\right)\right)$ contains a cycle of length $n - o(n)$ with high probability.

Exercise 2* Let X be the random variable counting the number of Hamiltonian cycles in $G(n,p)$. What is the best upper bound on the threshold for Hamiltonicity of $G(n,p)$ that you can prove by applying the second-moment method to X?

Exercise 3 Given a graph G with chromatic number $\chi(G) = 1000$, let $U \subseteq V(G)$ be a uniformly random set of vertices, and consider the induced subgraph $H = G[U]$. Prove that $\mathbb{P}(\chi(H) \leq 400) \leq \frac{1}{100}$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2020/hints/S5.html`.]

Exercise 4 By considering the vertex-exposure martingale applied to an appropriate choice of graph parameter, prove the existence of constants $c, C > 0$ such that if $p \geq Cn^{-2/3}$, then $\mathbb{P}\left(K_3 \not\subseteq G(n,p)\right) \leq \exp(-cn)$.

Exercise 5 A graph G is *vertex-Ramsey for* K_3 if every red-/blue-colouring of the *vertices* $V(G)$ gives rise to a monochromatic triangle. Let $p_0(n)$ be the threshold for $G(n,p)$ to be vertex-Ramsey for K_3.

(a) Prove that $p_0(n) \geq n^{-3/4}$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2020/hints/S5.html`.]

(b) Using the result from Exercise 4, show that $p_0(n) \leq n^{-2/3}$.

Exercise 6 Let $k_0 = k_0(n)$ be the smallest k for which $\binom{n}{k} 2^{-\binom{k}{2}} < 1$. As we have seen in a previous exercise sheet, $k_0 \sim 2\log n$, and for $G \sim G\left(n, \frac{1}{2}\right)$, the clique number satisfies $\omega(G) \in \{k_0 - 1, k_0\}$ with high probability.

Let $k_1 = k_0 - 4$. Using the techniques we saw in lecture,[1] one can show $\mathbb{P}(\omega(G) < k_1) \leq \exp\left(-\Omega(n^2/\log^8 n)\right)$.[2] Using this, prove the following facts about the chromatic number of G.

(a) With high probability, $\chi(G) = (1 + o(1))\frac{n}{2\log n}$.

(b) There is an interval $I_n \subseteq [n]$ of length $O\left(\frac{\sqrt{n}}{\log n}\right)$ such that $\mathbb{P}(\chi(G) \notin I_n) \leq \frac{1}{100}$.

[1] When we proved that, for $p \geq \frac{1}{\sqrt{3n}}$, we have $\mathbb{P}(K_3 \not\subseteq G(n,p)) \leq \exp\left(-\Omega(n^2p^2)\right)$.

[2] It would be a very good idea for you to prove this on your own!