

Exercise Sheet 6

Due date: 10:30, Jul 9th, to be submitted in Whiteboard.

Late submissions will receive as much credit as Lise Meitner did.

You should try to solve all of the exercises below, and clearly mark which two solutions you would like to be graded — each problem is worth 10 points. Starred exercises represent problems that may be a little tougher, should you wish to challenge yourself. In case you have difficulties submitting in Whiteboard, please send your solutions to probmethod@gmail.com.

Exercise 1 Pippinger's Theorem required three things of the hypergraph in question — that almost all vertices should have typical degrees, that the maximum degree should not be much larger than the typical degree, and that co-degrees should be small. Give counterexamples to show that the results of the theorem need not hold if any one of these conditions is not satisfied.

Exercise 2 Recall that a Latin square of order n is an $n \times n$ array filled with n different symbols, such that each symbol appears exactly once in each row and in each column. A transversal is a selection of cells of the array such that each row, column and symbol is represented at most once.

- (a) Show that, as $n \rightarrow \infty$, every Latin square of order n admits a transversal of size $(1 - o(1))n$.
- (b) Assuming the existence of Latin squares of every order n , prove the existence of 2 - $(3^r, 3, 1)$ designs for every $r \in \mathbb{N}$.

[Hint at <http://discretemath.imp.fu-berlin.de/DMIII-2020/hints/S6.html>.]

Exercise 3 Given some fixed $k \in \mathbb{N}$, the random k -uniform hypergraph $H^{(k)}(n, p)$ has vertices $[n]$, with each edge in $\binom{[n]}{k}$ appearing independently with probability p . Determine the threshold for the property of $H^{(k)}(n, p)$ having a matching of size $(1 - o(1))\frac{n}{k}$.

Exercise 4 When considering (k, t) -packings of $[n]$, we proved that the size of the largest such packing satisfies $m(n, k, t) \leq \frac{\binom{n}{t}}{\binom{k}{t}}$.

- (a) Show that if there is some prime power q such that $k = q + 1$ and $n = q^2 + q + 1$, then the upper bound on $m(n, k, 2)$ is tight.
- (b) Conversely, prove that if $\omega(\sqrt{n}) = k = o(n)$, then there is some constant $c \in (0, \frac{1}{2})$ such that $m(n, k, 2) \leq \frac{cn^2}{k^2}$.

[Hint at <http://discretemath.imp.fu-berlin.de/DMIII-2020/hints/S6.html>.]

Exercise 5* Going further than in Exercise 4(b) above, show that when $\omega(\sqrt{n}) = k = o(n)$, we have $m(n, k, 2) = o\left(\frac{n^2}{k^2}\right)$. What is the correct order of magnitude of $m(n, k, 2)$ in this case?

Exercise 6 You are tasked with organising a tournament¹ of four-player chess² involving the top n players of the game.⁴ In each round of the tournament, you will stage a number of games of four-player chess simultaneously — in particular, this means each of the players can play at most one game per round (they may receive a bye, in which case they do not play in that round).

In order to deliver a high-quality tournament that will keep spectators interested,⁵ you realise that two conditions should be met throughout the tournament:

- (i) no two games should feature the same set of four players, and
- (ii) in each round, at least 99% of the players should be playing.

Unfortunately, there is an added wrinkle. As is perhaps well-known, there are some very bitter rivalries within the chess community.⁷ To preempt any fistfights,⁸ you decide to allow each participant to provide a list of up to \sqrt{n} players that they would rather not play against. This gives rise to a third condition:

¹“Why in the world,” you ask, “would I ever be doing this?” I concede, that is a fair question. However, the “you” in this exercise is not the you that you are, but rather the you that you could be, in a parallel universe of sorts. Imagine, in this hypothetical world, that a deadly virus goes, as they say, viral, giving rise to a global pandemic. Seeking to protect their populace, governments all around the globe would wisely institute lockdowns to slow the spread of the disease. While this would no doubt help prevent medical systems from getting overwhelmed, humans are, for better or worse, social beings, and would thus be driven to the Internet in search of interpersonal interaction.

Thus, the silver lining on the pandemic stormcloud is that certain online businesses would thrive. Chief amongst those is the world of online chess, allowing people to pitch their wits in battle against both friends and strangers alike, all while practising responsible social distancing. Especially with the suspension of other forms of entertainment, interest in chess would skyrocket, with several online tournaments being held for a (literally) captive audience. It is of precisely such a tournament that you would, hypothetically speaking, be in charge.

²“Fine,” you grumble, “I can stretch my imagination to accept the outlandish assumptions of the previous footnote, but this ... this is a step too far! Everyone knows that chess is a two-player game!” Perhaps in your grandparents’ days, but this is 2020, and exciting new variants of chess abound! One such extension, which, unlike classical chess, is game-theoretically non-trivial, is four-player chess. Taking advantage of chess’s surge in popularity, one of the leading chess websites has taken great strides in promoting the four-player game.³

³<https://www.chess.com/4-player-chess>¹⁰

⁴This includes not only the leading 4-player chess players, such as Cha_Cha, icystun, Luke and Tom, but also some of the world’s best classical chess players, such as renowned grandmasters Anish, Hikaru and Vidit, and, of course, rising star and 25-time Dutch champion, Anna-Maja.

⁵For, after all, the tournament will be broadcast on a popular game-streaming website,⁶ and the longer the audience is watching, the more advertising revenue comes your way.

⁶<https://www.twitch.tv/directory/game/Chess>¹⁰

⁷For instance, it is unlikely that Anish will ever forgive Vidit his terrible betrayal in one of the first games of four-player chess to be publicised to the mainstream (chess) world.

⁸For that would stray into the domain of chessboxing, for which you do not have the rights.

(iii) nobody should play in a game with anyone on their list.

Subject to these requirements, your objective is to maximise the number of rounds in the tournament, so as to provide as much entertainment as possible (and, purely coincidentally, to make as much profit as you can). How many rounds can you guarantee, even in the worst case?⁹

⁹If, after all this, I have managed to inspire any curiosity for the wonderful yet peculiar world of four-player chess, then you will be delighted to know that (in our very own real world), the third world championship of the sport will soon be upon us. More details can be found here: <https://www.chess.com/clubs/forum/view/2020-third-online-world-4-player-chess-championship-teams-discussion>¹⁰

¹⁰Disclaimer: none of the mentioned companies have paid me for their promotion in this exercise. Should they wish to do so, I would be happy to negotiate terms.