Exercise Sheet 7

Due date: 10:30, Jul 16th, to be submitted in Whiteboard.
Late submissions will be as likely to be graded as $G(n, 0)$ is to contain a triangle.

You should try to solve all of the exercises below, and clearly mark which two solutions you would like to be graded — each problem is worth 10 points. Starred exercises represent problems that may be a little tougher, should you wish to challenge yourself. In case you have difficulties submitting in Whiteboard, please send your solutions to probmethod@gmail.com.

Exercises 1 and 2 test material from the last few weeks. Exercises 3, 4, 5 and 6 are review problems, and may cover content from earlier in the course.

Exercise 1 Following the outstanding success of your earlier four-player chess tournament, the powers that be order that you host another one. However, after your great efforts to popularise the game, there are now many more interested players, and so you can no longer afford to stage a tournament with anywhere close to $\Omega\left(\binom{n-1}{3}\right)$ rounds. Furthermore, you realised that when a pair of players plays multiple games together, they start coordinating their attacks, making the game unenjoyable for the other two players.

You therefore decide that the new tournament should satisfy the following guidelines.

1. In each round, all games will be played simultaneously, with every one of the n players playing at most one game in each round.
2. In each round, at least 99% of the players will be playing.
3. Every pair of players will have at most one game in common.

Show that, for every $\varepsilon > 0$, if n is sufficiently large then it is possible to design such a tournament with at least $(1-\varepsilon)\frac{n}{3}$ rounds.

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2020/hints/S7.html`.]

Exercise 2 In this exercise you are asked to locate the thresholds for certain properties in random graphs.

(a) A triangle factor of a graph G is a collection of vertex-disjoint triangles that cover all the vertices of G. If p_0 is the threshold for $G(3n, p)$ to have a triangle factor, show that $\Omega\left(n^{-2/3}\right) = p_0 = O\left(n^{-2/3}\log n\right)$.

(b) Given a graph H, the kth power H^k is obtained by adding an edge between every pair of vertices whose distance in H is at most k. Determine, up to a logarithmic factor, the threshold for $G(n, p)$ to contain the kth power of a Hamiltonian cycle, C_n^k.

Exercise 3 In this exercise we will take a closer look at the lower bound on the threshold for $G(3n, p)$ to contain a triangle factor.

(a) Let $p = o\left(n^{-1/2}\right)$, and let v be a fixed vertex of $G(3n, p)$. Show that the probability that v is not contained in any triangle is $\exp\left(-\Theta(n^2p^3)\right)$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2020/hints/S7.html`.]

(b) Deduce that $G(3n, p)$ with high probability does not contain a triangle factor when $p = o\left(n^{-2/3}(\log n)^{1/3}\right)$.

Exercise 4 Given disjoint sets U and W of vertices, of sizes n and m respectively, the random bipartite graph $G(n, m, p)$ is formed by adding an edge for each pair $(u, w) \in U \times W$ independently with probability p. Prove that the threshold for $G(n, n, p)$ to contain a perfect matching is $p_0 = \frac{\log n}{n}$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMIII-2020/hints/S7.html`.]

Exercise 5 Let H be a k-uniform hypergraph such that each vertex is in at most Δ edges. Show that if $7k\Delta \leq 2^k$, then the vertices of H can be two-coloured without creating a monochromatic edge.

Exercise 6 Prove that there is some constant $c > 0$ such that the rows of any $n \times n$ matrix $M \in \mathbb{R}^{n \times n}$ with pairwise-distinct entries can be reordered so that no column of M has an increasing subsequence of length at least $c\sqrt{n}$.