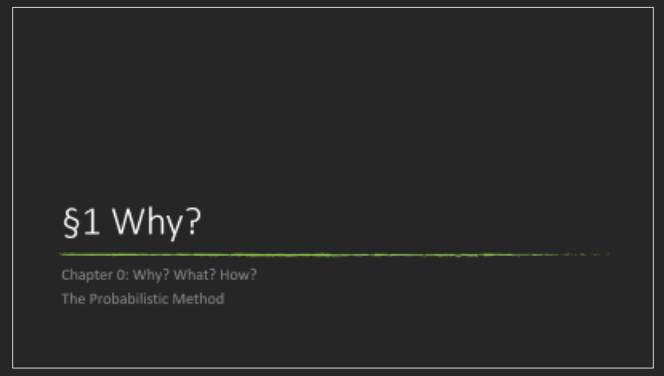
Chapter 0: Why? What? How?

The Probabilistic Method

Summer 2020

Freie Universität Berlin

Chapter Outline



Why should we learn the probabilistic method?

What is the probabilistic method?

How will we be learning this semester?

§1 Why?

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Existential Problems

Most important questions are existential in nature

Theorem 0.1.1 (Mantel, 1907)

The largest triangle-free graph on n vertices has $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges.

Lower bound: there is an n-vertex triangle-free graph with $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges

Upper bound: there is a triangle in any larger graph

Existential Problems - II

Riemann Hypothesis	Is there some $s \in \mathbb{C} \setminus (-2\mathbb{N})$ with $Re(s) \neq \frac{1}{2}$ for which $\zeta(s) = 0$?
Medicine	Is there a vaccine for the coronavirus?
Television	Does Game of Thrones' final season have any redeeming qualities?

Types of Solutions

Constructive solutions are ideal

- Not only prove the existence of the desired object, but show how to find it
- Especially important in real-world problems

Existential solutions still valuable

- Prove existence of the object without showing how to find it
- e.g.: in Mantel upper bound, don’t need to know how to find triangles in dense graphs to show that the complete balanced bipartite graph is optimal

Probabilistic method very powerful for proving existential results

- Often yield randomised constructions as well

§2 What?

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Formulating the Problem

Given

- A (finite) set Ω of objects
- A desired property $\mathcal{P}$

Goal

- Show there is some $\omega \in \Omega$ with the property $\mathcal{P}$

Ramsey Theory

- Ω: all graphs on n vertices
- $\mathcal{P}$: not having a clique or independent set on k vertices

A Probabilistic Approach

Idea

- Show that a *random* element $\omega \in \Omega$ could have the property $\mathcal{P}$

Formalism

- Define a positive probability measure $\mathbb{P}$ on the space $(\Omega, 2^{\Omega})$
- Existence question is equivalent to showing $\mathbb{P}(\omega \in \mathcal{P}) > 0$

Ramsey Theory

- Take $\mathbb{P}$ to be the uniform measure
  - Equivalent to each edge appearing independently with probability $\frac{1}{2}$
- Can show that if $n \approx \sqrt{2}^{k}$, then the probability of not having a clique or independent set of size k is positive

Aren't we just counting?

Ramsey Theory

- Total number of graphs:
- Number with a clique or independent set of size k:
- When $n \approx \sqrt{2}^k$, this is less than the total number of graphs

Advantages of thinking probabilistically

- Wide array of more advanced tools and methods
- Will introduce some of these in this course

§3 How?

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Cast and Crew

Lectures

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Exercises

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Online Resources

Whiteboard site

- Make sure you are enrolled at https://mycampus.imp.fu-berlin.de/portal/site/aed0cf99-5d64-48b7-bdb4-7f5764903675

Course website

- Further information available at http://discretemath.imp.fu-berlin.de/DMIII-2020/

Cisco Webex Meetings

- All course events will take place through Webex Meetings
- Links to the meetings can be found in Whiteboard

Schedule and Lectures

Timings

- Tuesdays and Thursdays, 10:30 – 12:00
- Every second week: Thursday lecture replaced by exercise class

Lectures

- Slides available on website in advance
- Annotated versions available afterwards
- Please share video, but keep audio muted unless speaking
  - Can also use chat to ask questions
- Occasional quizzes to check that everything is clear

Homework and Exercise Sessions

Problem sheets

- Posted online every two weeks, usually 1-2 weeks before due date
- Submit solutions (individually) to the Whiteboard site
  - Can be typed, or photos/scans of (neat) handwritten solutions
- Indicate which problems should be graded

Exercise sessions

- Will be split into two groups for the exercise sessions
- During the sessions, you will present and discuss solutions
  - Should sign up on the course website to present a problem
  - When presenting, can show your solution by sharing screen

Grading

Aktive Teilnahme

- Obtain $\geq 60\%$ of possible homework points
- Present ≥ 3 solutions
- Be an active participant in exercise classes

Exams

- Final grade comes from an oral exam
- To be offered in the summer – scheduled later

Any questions?