Chapter 2: Method of Alterations

The Probabilistic Method

Summer 2020

Freie Universität Berlin

Chapter Overview

- Introduce the method of alterations

§1 Ramsey Revisited

Chapter 2: Method of Alterations
The Probabilistic Method

§2 Dominating Sets

Chapter 2: Method of Alterations
The Probabilistic Method

§3 Dependent Random Choice

Chapter 2: Method of Alterations
The Probabilistic Method

§1 Ramsey Revisited

Chapter 2: Method of Alterations

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Asymmetric Ramsey Bounds

Definition 1.5.4 (Asymmetric Ramsey numbers)

Given $\ell, k \in \mathbb{N}$, $R(\ell, k)$ is the minimum n for which any n-vertex graph contains either a clique on ℓ vertices or an independent set on k vertices.

Obtained lower bounds by considering the random graph $G(n, p)$

Corollary 1.5.8

For fixed $\ell \in \mathbb{N}$ and $k \to \infty$, we have

$$\Omega\left(\left(\frac{k}{2\ln k}\right)^{\frac{\ell-1}{2}}\right) = R(\ell, k) = O\left(k^{\ell-1}\right).$$

Triangle-free Graphs

Corollary 1.5.8

For fixed $\ell \in \mathbb{N}$ and $k \to \infty$, we have

$$\Omega\left(\left(\frac{k}{2\ln k}\right)^{\frac{\ell-1}{2}}\right) = R(\ell, k) = O\left(k^{\ell-1}\right).$$

The case $\ell = 3$

- General lower bound gives $\Omega\left(\frac{k}{\ln k}\right)$
  - $R(3,k) \geq k$ is utterly trivial
  - Complete bipartite graph gives $R(3,k) \geq 2k-1$
- Can improve lower bound by more careful computation

Sharper Analysis

Theorem 1.5.7 ($\ell = 3$)

Given $k, n \in \mathbb{N}$ and $p \in [0,1]$, if

$$\binom{n}{3} p^3 + \binom{n}{k} (1-p)^{\binom{k}{2}} < 1,$$

then $R(3,k) > n$.

Better estimates

- $\binom{n}{3} p^3 \approx \frac{(np)^3}{6}$
- $\binom{n}{k} \approx 2^{H\left(\frac{k}{n}\right)n}$
- $(1-p)^{\binom{k}{2}} \approx e^{\frac{-pk^2}{2}}$

$\Rightarrow R(3,k) > ck$ for $c \approx 1.298$

Can We Go Further?

What happens for larger n?

- $\binom{n}{k} \geq \left(\frac{n}{k}\right)^k = e^{k \ln \frac{n}{k}}$
- $(1-p)^{\binom{k}{2}} > e^{-2p\binom{k}{2}} > e^{-pk^2}$
- $\Rightarrow$ need $p = \Omega\left(k^{-1} \ln \frac{n}{k}\right)$, otherwise $\binom{n}{k}(1-p)^{\binom{k}{2}}$ grows exponentially
- But then $\binom{n}{3}p^3 = \Theta\left((np)^3\right) = \Theta\left(\left(\frac{n}{k} \ln \frac{n}{k}\right)^3\right)$
  - Bigger than 1 if $n > Ck$ for some constant C

Lemma 2.1.1

There is some constant C such that, if $n > Ck$, then

$$\binom{n}{3}p^3 + \binom{n}{k}(1-p)^{\binom{k}{2}} > 1.$$

Reinterpreting the Proof

Proof we saw

- $\mathbb{P}(\{G(n,p) \text{ not Ramsey}\}) \leq \binom{n}{3}p^3 + \binom{n}{k}(1-p)^{\binom{k}{2}}$
- If this is less than 1, we get a Ramsey graph with positive probability
- If this is more than 1, we get no useful information

Linearity of expectation

- $\binom{n}{3}p^3$: expected number of triangles in $G(n,p)$
- $\binom{n}{k}(1-p)^{\binom{k}{2}}$: expected number of independent sets of size k in $G(n,p)$
- $\Rightarrow \binom{n}{3}p^3 + \binom{n}{k}(1-p)^{\binom{k}{2}}$ is the expected number of bad subgraphs
- If this is less than 1, then with positive probability we have no bad subgraphs
- $\Rightarrow$ we get a Ramsey graph

Shades of Grey

Great expectations

- What does $\mathbb{E}[\#\text{ bad subgraphs}] \geq 1$ mean?
- Do we have to have bad subgraphs?
  - Not necessarily; see Chapter 3 for details
- Gives some guarantee of goodness
  - There is a graph with at most $\binom{n}{3}p^3 + \binom{n}{k}(1-p)^{\binom{k}{2}}$ bad subgraphs
  - If this is small, perhaps we can fix it

Method of Alterations

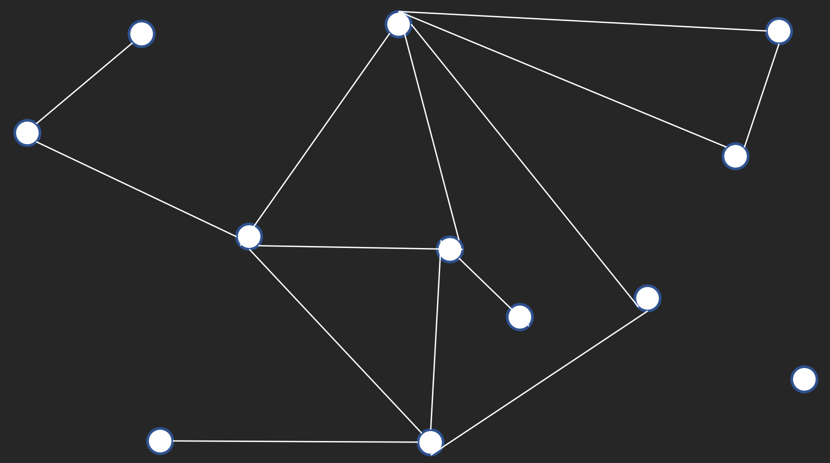
Goal: existence of an object with property $\mathcal{P}$

1. Show random object is with positive probability *close* to having $\mathcal{P}$
2. Make deterministic changes to the random object to achieve $\mathcal{P}$

Graph Surgery

Given

- Graph with few triangles/large independent sets



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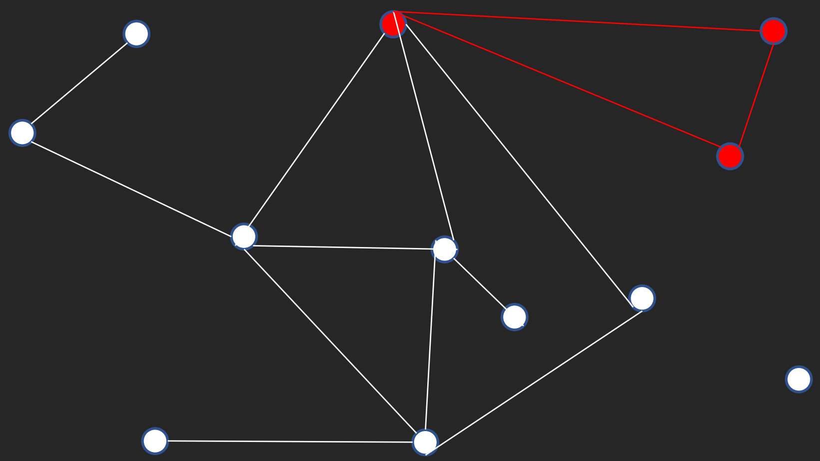

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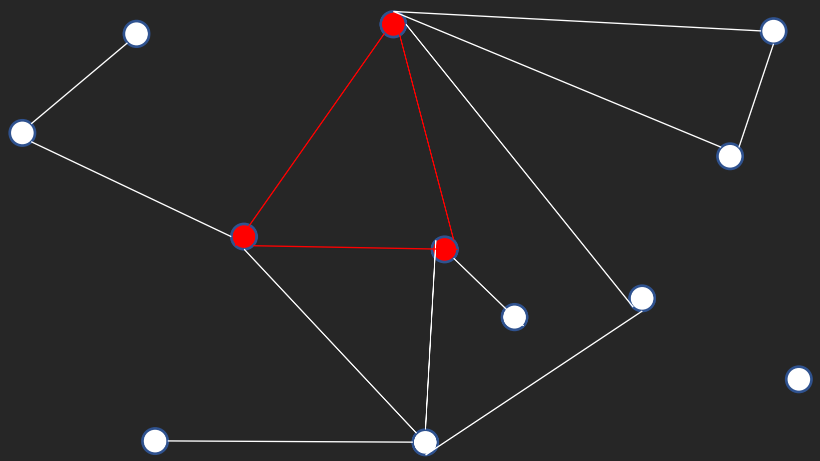

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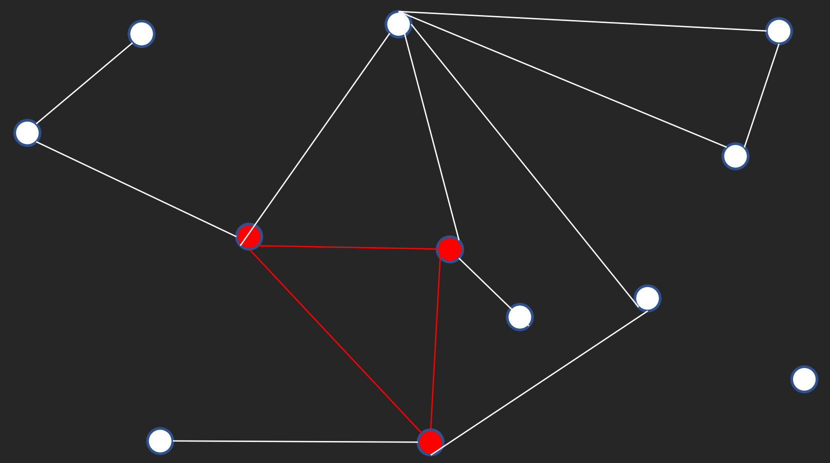

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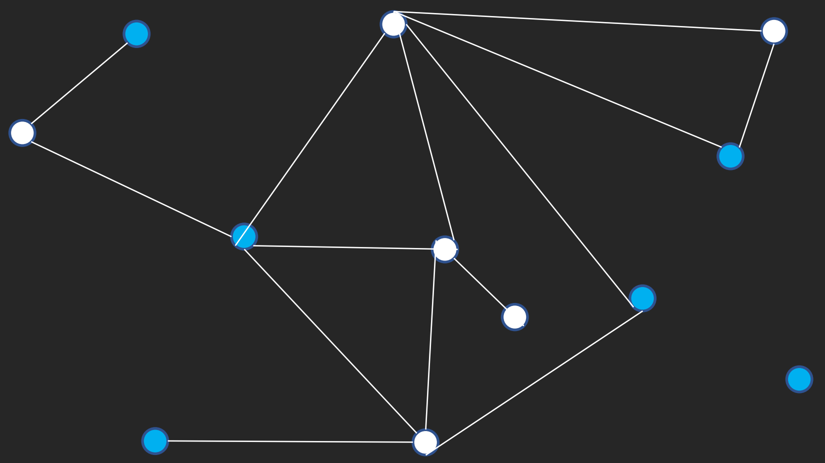

Graph Surgery

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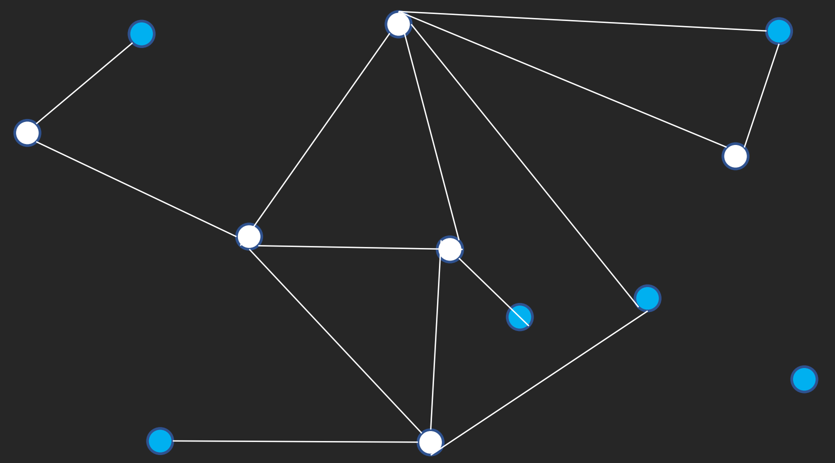


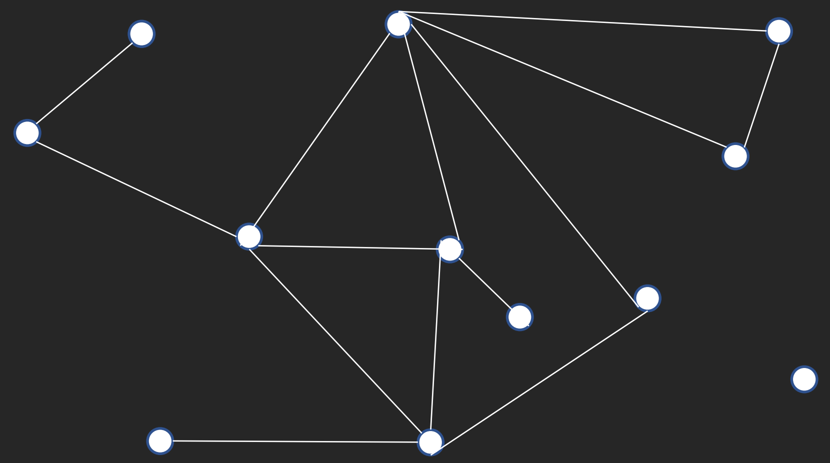
Graph Surgery

Given

- Graph with few triangles/large independent sets

Goal

- Edit graph to obtain a Ramsey graph



Idea: remove an edge from each triangle

Graph Surgery

Given

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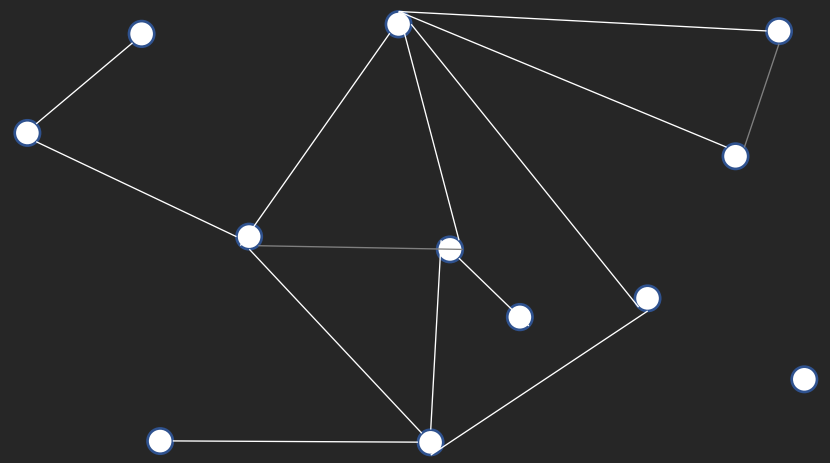

Idea: remove an edge from each triangle

Graph Surgery

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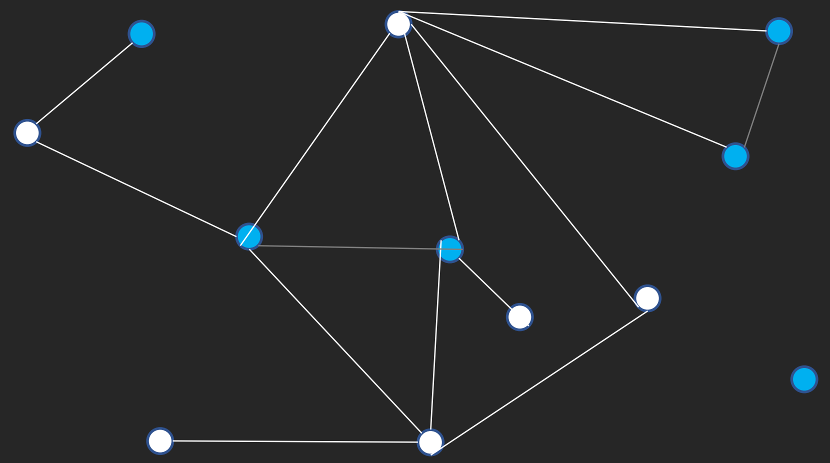
- Edit graph to obtain a Ramsey graph



Idea: remove an edge from each triangle

Problem: creates new independent sets

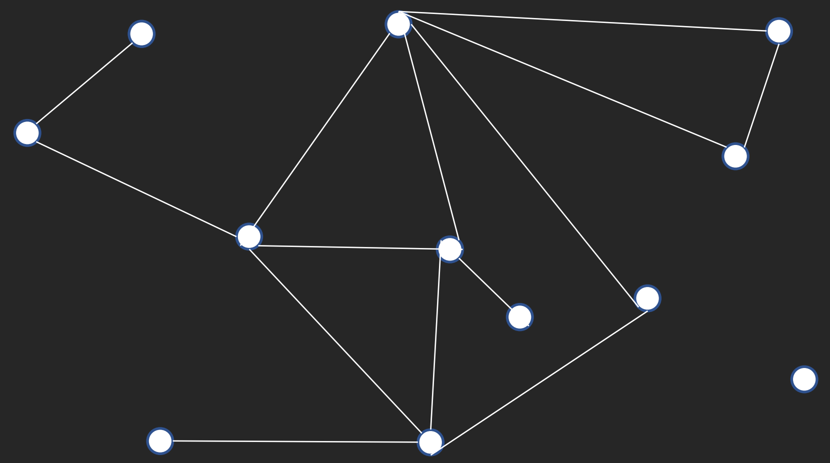
Graph Surgery

Given

- Graph with few triangles/large independent sets

Goal

- Edit graph to obtain a Ramsey graph



Solution: remove a *vertex* from each triangle/independent set

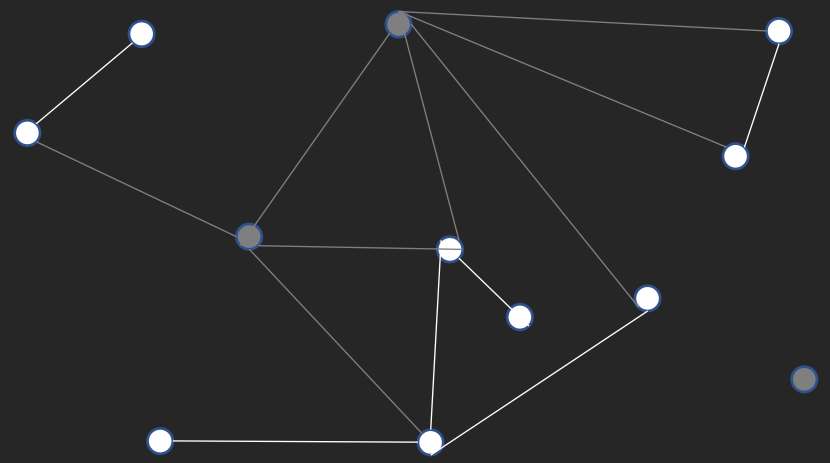
Graph Surgery

Given

- Graph with few triangles/large independent sets

Goal

- Edit graph to obtain a Ramsey graph



Solution: remove a *vertex* from each triangle/independent set

Result: a Ramsey graph, albeit on fewer vertices

An Altered Theorem

Theorem 2.1.2

For every $n, \ell, k \in \mathbb{N}$ and $p \in [0,1]$, we have

$$R(\ell, k) > n - \binom{n}{\ell} p^{\binom{\ell}{2}} - \binom{n}{k} (1-p)^{\binom{k}{2}}.$$

Proof

- Let $G \sim G(n,p)$
- $\mu := \binom{n}{\ell} p^{\binom{\ell}{2}} + \binom{n}{k}(1-p)^{\binom{k}{2}}$ is the expected number of K_ℓ and $\overline{K_k}$
- $\Rightarrow$ there is an n-vertex graph with at most μ bad subgraphs
- Delete one vertex from each bad subgraph
- Obtain a Ramsey subgraph on at least $n - \mu$ vertices $\blacksquare$

$R(3,k)$: A New Bound

Theorem 2.1.2 ($\ell = 3$)

For every $n, k \in \mathbb{N}$ and $p \in [0,1]$, we have

$$R(3,k) > n - \binom{n}{3} p^3 - \binom{n}{k} (1-p)^{\binom{k}{2}}.$$

Goal

- Choose n, p to maximise $n - \binom{n}{3}p^3 - \binom{n}{k}(1-p)^{\binom{k}{2}}$

Choosing p

- Small p makes the second term small
- Recall: need $p = \Omega\left(k^{-1} \ln \frac{n}{k}\right)$, otherwise third term exponentially large
- When p is this large, third term exponentially small – insignificant

$R(3,k)$: A New Bound

Recall

- Maximising $n - \binom{n}{3}p^3 - \binom{n}{k}(1-p)^{\binom{k}{2}}$
- Take $p = \Theta\left(k^{-1}\ln\frac{n}{k}\right)$

Choosing n

- Want to maximise $n - \Theta\left(\left(\frac{n}{k}\ln\frac{n}{k}\right)^3\right)$
- At maximum: $\left(\frac{n}{k}\ln\frac{n}{k}\right)^3 = \Theta(n)$
- $\Rightarrow n = \Theta\left(\left(\frac{k}{\ln\frac{n}{k}}\right)^{\frac{3}{2}}\right) = \Theta\left(\left(\frac{k}{\ln k}\right)^{\frac{3}{2}}\right)$

Where We Stand

Corollary 2.1.3

As $k \to \infty$, we have

$$\mathrm{R}(3, \mathrm{k}) = \Omega\left(\left(\frac{k}{\ln k}\right)^{\frac{3}{2}}\right).$$

Lower bound

- Superlinear lower bound
- Beats Turán

Upper bound

- Erdős-Szekeres: $R(3, k) = O(k^2)$
- Can we narrow the gap? Stay tuned!

Any questions?

§2 Dominating Sets

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The Probabilistic Method

BER: A Modern Tragicomedy

Sep 2006	Berlin-Brandenburg Airport to open Oct 2011
Jun 2010	Opening postponed to Jun 2012
May 2012	Fire detection systems do not work!

Solution

- Hire people to stand around the airport looking for signs of fire

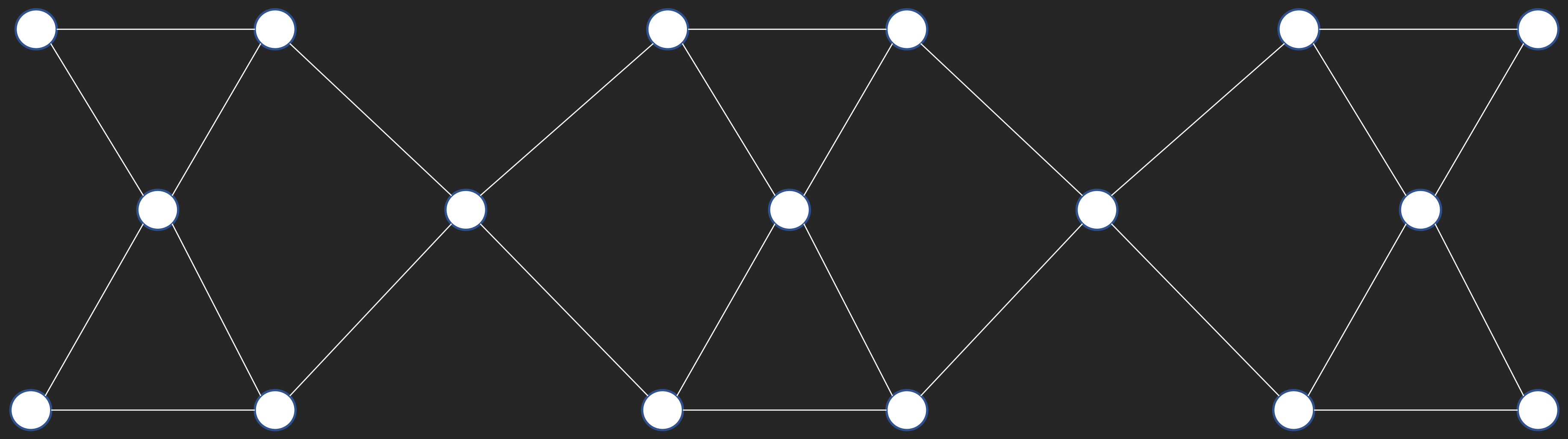
Problem

- Already overbudget
- $\Rightarrow$ want to hire as few people as possible

Combinatorics to the Rescue

The airport is a graph

- Vertices: areas where fire could break out
- Edges: lines of sight between areas



Objective

- Find a set of vertices that “see” all other vertices

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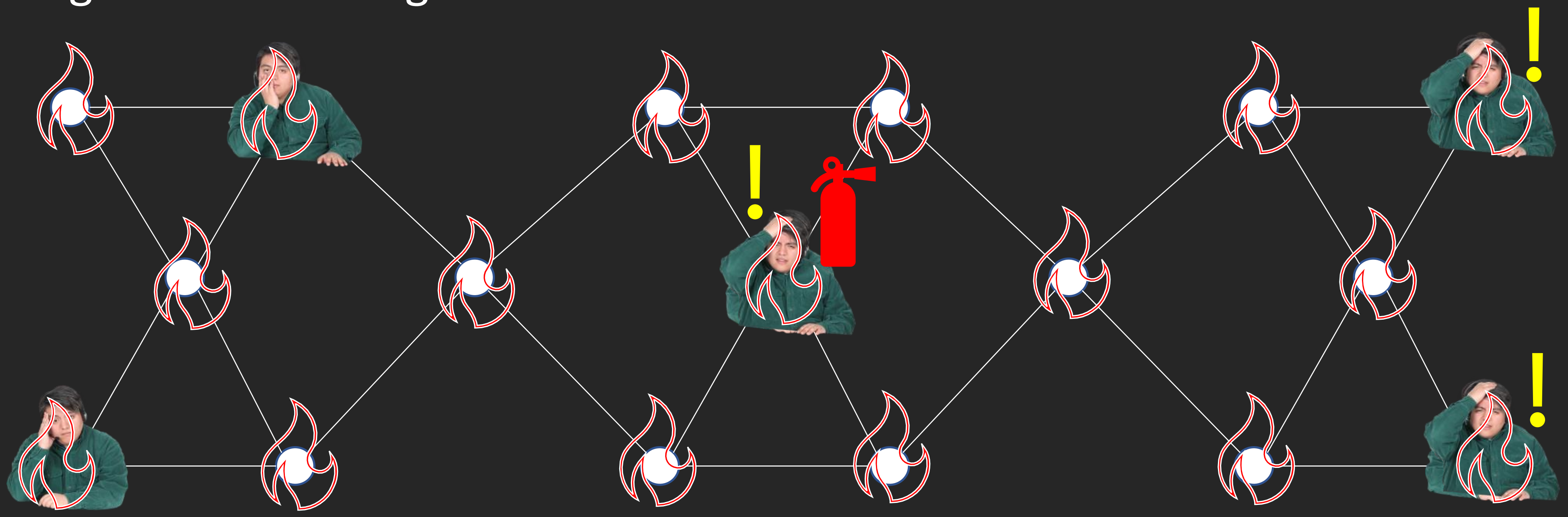

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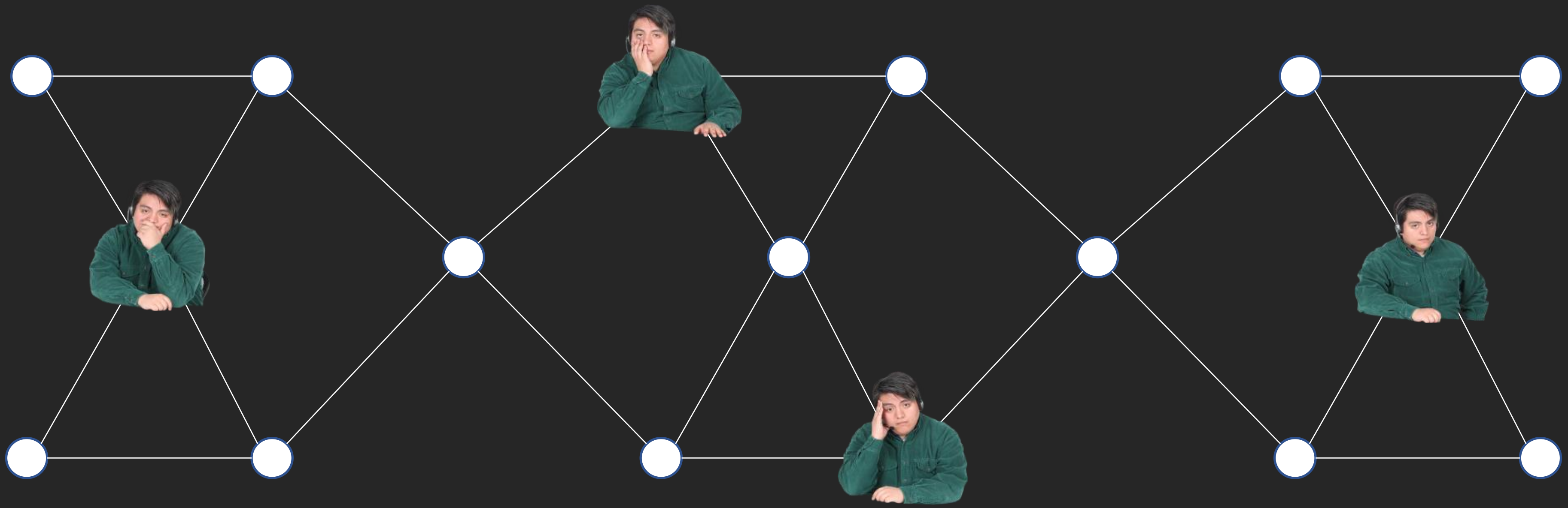

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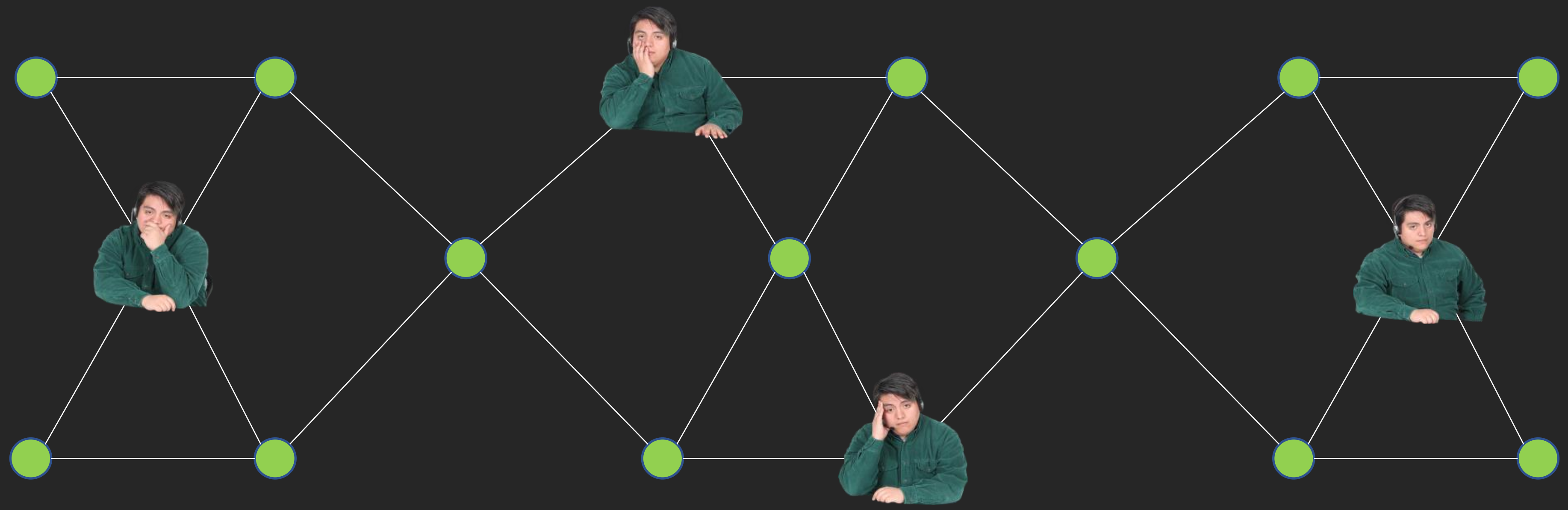

Objective

- Find a set of vertices that “see” all other vertices

Small Dominating Sets

Definition 2.2.1

Given a graph $G = (V, E)$, a set $S \subseteq V$ of vertices is a *dominating set* if, for every $v \in V \setminus S$, there is some $s \in S$ with $\{s, v\} \in E$.

Extremal problem

- How large can the smallest dominating set of an n-vertex graph G be?

Answer

- n (!)
- Isolated vertices must be in any dominating set

Avoiding trivialities

- What if we require G to have minimum degree δ?

Do Random Sets Dominate?

Problem

Given G on n vertices with $\delta(G) \geq \delta$, how large can its smallest dominating set be?

Random set

- Let $S \subseteq V$ be a random set
- $v \in S$ with probability p, independently

Undominated vertices

- For $u \in V$, define the event $E_u = \{u \text{ not dominated by } S\}$
- For E_u to hold, need:
  - $u \notin S$
  - $v \notin S$ for all neighbours v of u
- $\Rightarrow \mathbb{P}(E_u) = (1-p)^{d(u)+1}$

Calculations Continued

Failure probability

- $\{S \text{ not dominating}\} = \cup_{u \in V} E_u$
- $\mathbb{P}(\cup_{u \in V} E_u) < \sum_{u \in V} \mathbb{P}(E_u) = \sum_{u \in V} (1-p)^{d(u)+1}$
- $\sum_{u \in V} (1-p)^{d(u)+1} \leq n(1-p)^{\delta+1} \leq ne^{-p(\delta+1)}$
- $\Rightarrow$ if $p = \frac{\ln n}{\delta+1}$, then $\mathbb{P}(S \text{ not dominating}) < 1$
- $\Rightarrow S$ is dominating with positive probability

Size of the dominating set

- $|S| \sim \mathrm{Bin}(n, p)$
- $\Rightarrow \mathbb{E}[|S|] = np = \frac{n \ln n}{\delta+1}$
- $\Rightarrow$ with positive probability, $|S| \leq \frac{n \ln n}{\delta+1}$

Putting Things Together

Concurrence of events

- Want events $\{S \text{ is dominating}\}$ and $\{S \text{ is small}\}$ to hold simultaneously
- Suffices to have $\mathbb{P}(S \text{ not dominating}), \mathbb{P}(S \text{ large}) < \frac{1}{2}$

Non-domination

- $\mathbb{P}(S \text{ not dominating}) < ne^{-p(\delta+1)} = \frac{1}{2}$ if $p = \frac{\ln 2n}{\delta+1}$

Large sets

- Binomial distribution $\Rightarrow \mathbb{P}(|S| > (n+1)p) \leq \frac{1}{2}$

Proposition 2.2.3

Let G be an n-vertex graph with $\delta(G) \geq \delta \geq \ln 2n$. Then G has a dominating set $S \subseteq V(G)$ with $|S| \leq \frac{(n+1)\ln 2n}{\delta+1}$.

Altering Our Approach

Reduced requirements

- Need large probability p for the random set S to be dominating
- What if we instead only want it to be *close* to dominating?
- Set S should dominate *most* vertices of G

Undominated vertices

- Given a graph G, set of vertices $\mathrm{T} \subseteq V(G)$
- Let $U(T) = \{v \in V \setminus T : N(v) \cap T = \emptyset\}$ be the vertices not dominated by T

Observation

For any $\mathrm{T} \subseteq V$, the set $\mathrm{T} \cup U(T)$ is a dominating set.

Altering Our Results

Theorem 2.2.4

Let G be an n-vertex graph with $\delta(G) \geq \delta$, and let $p \in [0,1]$. Then G has a dominating set $S \subseteq V(G)$ with $|S| \leq np + ne^{-p(\delta+1)}$.

Proof

- Let T be a random set of vertices, chosen independently with probability p
- $\Rightarrow \mathbb{E}[|T|] = np$
- Recall: $\mathbb{P}(u \text{ not dominated by } T) = (1-p)^{d(u)+1} \leq e^{-p(\delta+1)}$
- Linearity of expectation $\Rightarrow \mathbb{E}[|U(T)|] \leq ne^{-p(\delta+1)}$
- Let $S = T \cup U(T)$
  - S is a dominating set
- $\mathbb{E}[|S|] = \mathbb{E}[|T \cup U(T)|] = \mathbb{E}[|T|] + \mathbb{E}[|U(T)|] \leq np + ne^{-p(\delta+1)}$
- $\Rightarrow$ existence of a dominating set of at most this size ■

Running the Numbers

Goal

- Minimise $np + ne^{-p(\delta+1)} = n\left(p + e^{-p(\delta+1)}\right)$

A little calculus

- Let $f(p) = p + e^{-p(\delta+1)}$
- $f'(p) = 1 - (\delta + 1)e^{-p(\delta+1)}$
- $f'(p_0) = 0 \Leftrightarrow p_0 = \frac{\ln(\delta+1)}{\delta+1}$
- $f(p_0) = \frac{\ln(\delta+1)+1}{\delta+1}$

Corollary 2.2.5

Let G be an n-vertex graph with $\delta(G) \geq \delta$. Then G has a dominating set $S \subseteq V(G)$ with $|S| \leq \left(\frac{\ln(\delta+1)+1}{\delta+1}\right) n$.

BER: Epilogue

May 2012	Firewatch plan rejected, opening set for Mar 2013
Sep 2012	Opening postponed further to Oct 2013
2013-2019	Series of delays, no new opening date set
Apr 2020	Building authority approval! Opening 31 Oct 2020
Total delays	3072 days (and counting?)
Original budget	€2.3 billion
Actual cost	€7.3 billion (and counting?)

Any questions?

§3 Dependent Random Choice

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Turán Numbers

Definition 2.3.1

Given a graph H and $n \in \mathbb{N}$, the *Turán number* $\mathrm{ex}(n, H)$ is the maximum number of edges in an H-free n-vertex graph.

Theorem 1.5.6 (Turán, 1941)

For $\ell \geq 2$, $\mathrm{ex}(n, K_\ell) = \left(1 - \frac{1}{\ell-1} + o(1)\right)\binom{n}{2}$.

Theorem 2.3.2 (Erdős-Stone-Simonovits, 1966)

For any graph H, $\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)\binom{n}{2}$.

Bipartite Turán Numbers

Theorem 2.3.2 (Erdős-Stone-Simonovits, 1966)

For any graph H, $\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)\binom{n}{2}$.

- Determines $\mathrm{ex}(n, H)$ asymptotically when $\chi(H) \geq 3$
- H bipartite: only shows $\mathrm{ex}(n, H) = o(n^2)$

Theorem 2.3.3 (Kővári-Sós-Turán, 1954)

If H is bipartite with at most t vertices in one part, then

$$\mathrm{ex}(n, H) = O\left(n^{2-1/t}\right).$$

Tightness of Kővári-Sós-Turán

Theorem 2.3.3 (Kővári-Sós-Turán, 1954)

If H is bipartite with at most t vertices in one part, then

$$\mathrm{ex}(n, H) = O\left(n^{2-1/t}\right).$$

Complete bipartite graphs

- Tight for $H = K_{t,s}$ when $s > (t-1)!$ [Alon-Rónyai-Szabó, 1999]

Even cycles

- Far from tight for cycles
- $\mathrm{ex}(n, C_{2k}) = O(n^{1+1/k})$ [Bondy-Simonovits, 1974]

General graphs

- Can we find a sharper general bound?

Dependent Random Choice

Embedding H

- Need to show any sufficiently dense graph G must contain a copy of H
- We know nothing about G apart from its density
- This is enough to extract some structure

Lemma 2.3.4 (Dependent Random Choice)

Let $a, d, m, n, t \in \mathbb{N}$. Let G be an n-vertex graph with average degree d. If there is some $s \in \mathbb{N}$ with

$$\frac{d^s}{n^{s-1}} - \binom{n}{t}\left(\frac{m}{n}\right)^s \geq a,$$

then G contains a subset A of at least a vertices, any t of which have more than m common neighbours.

A Turánnical Application

Theorem 2.3.5 (Alon-Krivelevich-Sudakov, 2003)

Let H be a bipartite graph with maximum degree t in one part. Then

$$ex(n, H) = O\left(n^{2-1/t}\right).$$

Kővári-Sós-Turán

- Immediate consequence of the above theorem
- Same examples show bound can be tight

Wider class of graphs

- Gives reasonable bounds for graphs of arbitrary order
- e.g. even subdivisions F_{sub} of a graph F
  - Each edge of F replaced by an even path
- Can apply Theorem 2.3.5 with $t = 2 \Rightarrow ex(n, F_{sub}) = O(n^{3/2})$

Setting Up the Proof

Theorem 2.3.5 (Alon-Krivelevich-Sudakov, 2003)

Let H be a bipartite graph with maximum degree t in one part. Then

$$ex(n, H) = O\left(n^{2-1/t}\right).$$

Given

- Bipartite H with vertex classes $U \cup W$
- Maximum degree in W is t

Objective

- Given n-vertex graph G with $e(G) \geq \Omega(n^{2-1/t})$
- Need to show $H \subseteq G$

Applying Dependent Random Choice

Lemma 2.3.4 (Dependent Random Choice)

Let $a, d, m, n, t \in \mathbb{N}$. Let G be an n-vertex graph with average degree d. If there is some $s \in \mathbb{N}$ with

$$\frac{d^s}{n^{s-1}} - \binom{n}{t}\left(\frac{m}{n}\right)^s \geq a,$$

then G contains a subset A of at least a vertices, any t of which have more than m common neighbours.

Idea

- Embed U in A arbitrarily
- Each $w \in W$ has at most t neighbours in U
- Corresponding set of t vertices in A has at least m common neighbours in G
- May have used some on earlier vertices, but if $m \geq v(H)$, one is free to embed w
- $\Rightarrow$ can embed W one vertex at a time

A Little Arithmetic

Target

- $\frac{d^s}{n^{s-1}} - \binom{n}{t}\left(\frac{m}{n}\right)^s \geq a$ where
  - $a = |U| \leq v(H) =: h$
  - $m = h$
  - $d = C_H n^{1-1/t}$ for some constant C_H we can choose
  - we can choose $s \in \mathbb{N}$

Simplify

- $\binom{n}{t} \leq n^t$
- Sufficient to have $C_H^s n^{1-s/t} - h^s n^{t-s} \geq h$
- $\Rightarrow$ need to take $s = t$
- Sufficient to have $C_H^t \geq h^t + h$
- Satisfied by taking $C_H = 2^{1/t} h$, completing the proof ∎

Proving Dependent Random Choice

Lemma 2.3.4 (Dependent Random Choice)

Let $a, d, m, n, t \in \mathbb{N}$. Let G be an n-vertex graph with average degree d. If there is some $s \in \mathbb{N}$ with

$$\frac{d^s}{n^{s-1}} - \binom{n}{t}\left(\frac{m}{n}\right)^s \geq a,$$

then G contains a subset A of at least a vertices, any t of which have at least m common neighbours.

Does a random set work for A?

- No - G could be bipartite
- Then a random set will intersect both parts
- Subsets meeting both parts have no common neighbours

An Indirect Selection

Idea

- We choose a small random set S of vertices
- Let their *common neighbourhood* B be our candidate for A

Intuition

- G has large average degree d
  - $\Rightarrow S$ should have many common neighbours
- If a set of vertices has few neighbours, unlikely that S was chosen from them
  - $\Rightarrow$ will not see these t vertices in B
- Can expect t-subsets of B to have large common neighbourhood

Fleshing Out the Details

Choosing S

- Sample s vertices from $V(G)$, independently (with repetition!)
- Let S be the set of vertices selected

Common neighbourhood B

- Let $B = \{v \in V(G) \colon \forall s \in S, \{v, s\} \in E(G)\}$
- For $v \in B$, all vertices in S had to be neighbours of v
- $\Rightarrow \mathbb{P}(v \in B) = \left(\frac{d(v)}{n}\right)^s$
- $\Rightarrow \mathbb{E}[|B|] = \sum_v \left(\frac{d(v)}{n}\right)^s = n^{-s} \sum_v d(v)^s$
- $x \mapsto x^s$ is a convex function
- $\Rightarrow \mathbb{E}[|B|] = n^{-s} \sum_v d(v)^s \geq n^{1-s} \left(\frac{\sum_v d(v)}{n}\right)^s = \frac{d^s}{n^{s-1}}$

Fixing the Set

Bad subsets

- Let T be a set of t vertices with at most m common neighbours
- To have $T \subseteq B$, need to select S from these common neighbours
- $\Rightarrow \mathbb{P}(T \subseteq B) \leq \left(\frac{m}{n}\right)^s$
- Linearity of expectation $\Rightarrow \mathbb{E}[\#\text{ bad subsets}] \leq \binom{n}{t}\left(\frac{m}{n}\right)^s$

Alteration

- Remove one vertex from each bad subset
- Let A be the remaining set
  - Every t-subset of A has more than m common neighbours
- $\mathbb{E}[|A|] \geq \mathbb{E}[|B|] - \mathbb{E}[\#\text{ bad subsets}] \geq \frac{d^s}{n^{s-1}} - \binom{n}{t}\left(\frac{m}{n}\right)^s \geq a$
- $\Rightarrow$ there exists such a set A of size at least a ∎

Any questions?