Chapter 4: The Lovász Local Lemma

The Probabilistic Method

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Chapter Overview

- Introduce the Lovász Local Lemma and some variants
- Survey some applications, including to $R(3, k)$

§1 Introducing the Lemma

Chapter 4: The Lovász Local Lemma

The Probabilistic Method

Avoiding Bad Events

Second moment set-up

- Have a collection of good events $E_1, E_2, \dots, E_m$
  - e.g.: $E_i = \{i\text{th copy of } H \text{ appears in } G(n,p)\}$
- Goal: show that with positive probability, at least one event occurs
- Usually show this happens with probability $1 - o(1)$

Opposite situation

- Have a collection of bad events $E_1, E_2, \dots, E_m$
  - e.g.: $E_i = \{i\text{th clause in } k\text{−SAT formula not satisfied}\}$
- Goal: show that with positive probability, none of these events occur
  - i.e.: $\mathbb{P}(\cap_i E_i^c) > 0$
- Union bound: $\mathbb{P}(\cap_i E_i^c) = 1 - \mathbb{P}(\cup_i E_i) \geq 1 - \sum_i \mathbb{P}(E_i)$
  - Tight when events E_i are disjoint
  - In general, need either m or $\mathbb{P}(E_i)$ to be small enough for effective bounds

Independence to the Rescue

Independent events

- If the events $E_1, E_2, \dots, E_m$ are mutually independent, we are in business
- $\mathbb{P}(\cap_i E_i^c) = \prod_i \mathbb{P}(E_i^c) = \prod_i \big(1 - \mathbb{P}(E_i)\big)$
- Might tend to zero, but is still positive (provided $\mathbb{P}(E_i) < 1$ for all i)
  - Doesn't matter how many bad events there are, or how likely they are

A real-world example

- Work for the Bundesdruckerei
  - Job: printing m passports
- Bad event: $E_i = \{\text{misprint in the } i\text{th passport}\}$
  - Say $\mathbb{P}(E_i) = \frac{1}{2}$ for each i
- $\mathbb{P}(\cap_i E_i^c) = \left(\frac{1}{2}\right)^m > 0$
  - $\Rightarrow$ it is possible to have a successful day

The Struggle for Independence

Do we need independence?

- In practice, true independence of events is rare
- Could hope to replace it with something weaker
  - *Most* events being independent? Pairwise independence?

We might

- Bundesdruckerei example: suppose our passport printer is odd
  - Never makes an even number of misprints
- Same marginal distributions
  - $\mathbb{P}(E_i) = \frac{1}{2}$ for all i
- Almost complete independence
  - Any $m - 1$ of the m events are mutually independent
- However, $\mathbb{P}(\cap_i E_i^c) \leq \mathbb{P}(\#\text{ misprints even}) = 0$

Lovász to the Rescue

The Bundesdruckerei problem

- $\mathbb{P}(E_i) = \frac{1}{2}$ is a large probability for the bad event
- If $\mathbb{P}(E_i) < \frac{1}{2}$, then we lose even pairwise independence
  - $\mathbb{P}(E_i | E_j) < \mathbb{P}(E_i)$

The good news

- Suppose the bad events
  - are independent of most other events
  - occur with reasonably small probability
- Lovász Local Lemma $\Rightarrow$ events behave as if independent
- Can show that with positive probability none occur

The Local Lemma – Symmetric Setting

Theorem 4.1.1 (Symmetric Lovász Local Lemma; Erdős-Lovász, 1975)

Let $E_1, E_2, \dots, E_m$ be events such that each event E_i is mutually independent of all but at most d of the other events, and $\mathbb{P}(E_i) \leq p$ for all i. If $ep(d+1) \leq 1$, then $\mathbb{P}(\cap_i E_i^c) > 0$.

"Local" Lemma

- Bound on p independent of number of events (global property)
- Only depends on number of dependencies (local property)

Conclusion

- Only assert that with positive probability, none of the events occur
- This probability can depend on the number of events

Re-restricted k-SAT

Recall

- Any k-SAT formula with fewer than 2^k clauses is satisfiable
- Bound is best possible: take formula with all clauses on k variables

Restricted k-SAT

- Previously: each k-set of variables appears in at most one clause
- What if we bound individual variable appearances instead?

Theorem 4.1.2

Any k-SAT formula in which each variable appears at most $\frac{2^k}{ek}$ times is satisfiable.

- Applies to k-SAT formulae with any number of clauses!

Proof by Local Lemma

Theorem 4.1.2

Any k-SAT formula in which each variable appears at most $\frac{2^k}{ek}$ times is satisfiable.

Proof

- Set each variable to true/false independently with probability $\frac{1}{2}$
- Events
  - $E_i = \{i\text{th clause not satisfied}\}$
  - $\mathbb{P}(E_i) = p := 2^{-k}$ for all i
- Dependencies
  - A clause is independent of any clauses with disjoint sets of variables
  - Clause has k variables, each in $\leq \frac{2^k}{ek} - 1$ other clauses
  - $\Rightarrow$ each event independent of all but $d := \frac{2^k}{e} - 1$ other events
- $ep(d+1) = 1 \Rightarrow \mathbb{P}(\cap_i E_i) > 0 \Rightarrow$ formula is satisfiable! ■

Recalling Ramsey

Theorem 1.5.2 (Erdős, 1947)

As $k \to \infty$, we have

$$R(k) \geq \left(\frac{1}{e\sqrt{2}} + o(1)\right) k\sqrt{2}^k.$$

- Disjoint sets of edges are independent
  - Can improve bound with the local lemma

Theorem 4.1.3

As $k \to \infty$, we have

$$R(k) \geq \left(\frac{\sqrt{2}}{e} + o(1)\right) k\sqrt{2}^k.$$

Setting Up the Proof

Events

- We take $G \sim G\left(n, \frac{1}{2}\right)$ as before
- For $I \in \binom{[n]}{k}$, event $E_I = \{G[I] \cong K_k \text{ or } K_k^c\}$
- $\mathbb{P}(E_I) = 2^{1-\binom{k}{2}}$ for all I

Dependencies

- E_I independent of all events with disjoint edge-sets
- $\Rightarrow E_I$ depends on at most $\binom{k}{2}\left(\binom{n-2}{\mathrm{k}-2} - 1\right)$ other events
- $\Rightarrow d + 1 \leq \binom{k}{2}\binom{n-2}{\mathrm{k}-2}$

Lovász Local Lemma

- $\Rightarrow$ suffices to show $e2^{1-\binom{k}{2}}\binom{k}{2}\binom{n-2}{k-2} \leq 1$

Running the Calculations

Estimates

- $\binom{k}{2} \leq \frac{k^2}{2}$
- $\binom{n-2}{k-2} \leq \frac{k^2}{n^2}\binom{n}{k} \leq \frac{k^2}{n^2}\left(\frac{ne}{k}\right)^k$

Bounding n

- $\Rightarrow e2^{1-\binom{k}{2}}\binom{k}{2}\binom{n-2}{k-2} \leq \frac{e2^{-\binom{k}{2}}k^4}{n^2}\left(\frac{ne}{k}\right)^k = \frac{ek^4}{n^2}\left(\frac{ne\sqrt{2}}{k\sqrt{2}^k}\right)^k$
- If $n = \frac{1}{e\sqrt{2}}k\sqrt{2}^k$, parenthetical term is 1
- Leading coefficient is then $e^2k^4 2^{1-k}$
  - $\Rightarrow$ can afford for the parenthetical term to be $2 + o(1)$
- $\Rightarrow$ can take $n = \left(\frac{\sqrt{2}}{e} + o(1)\right)k\sqrt{2}^k$ ■

Any questions?

§2 The Ramsey Number $R(3, k)$

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Returning to $R(3,k)$

Corollary 2.1.3

As $k \to \infty$, we have

$$\Omega\left(\left(\frac{k}{\ln k}\right)^{\frac{3}{2}}\right) = R(3,k) = O(k^2).$$

Lower bound

- Proven using $G \sim G(n,p)$ and alterations

Limited dependence

- Again, disjoint sets of edges are independent
- What does the Local Lemma give?

Analysing the Events

Two classes of events

- For $I \in \binom{[n]}{3}$, let $E_I = \{G[I] \cong K_3\}$
- For $J \in \binom{[n]}{k}$, let $F_J = \{G[J] \cong K_k^c\}$

Probabilities

- For each $I \in \binom{[n]}{3}$, $p_1 \coloneqq \mathbb{P}(E_I) = p^3$
- For each $J \in \binom{[n]}{k}$, $p_2 \coloneqq \mathbb{P}(F_J) = (1-p)^{\binom{k}{2}} \approx e^{-p\binom{k}{2}}$
- $\Rightarrow$ in Lovász Local Lemma, should take $p' = \max\{p_1, p_2\}$
- $\Rightarrow$ optimal to have $p_1 = p_2$
  - $\Rightarrow p \approx \frac{12 \ln k}{k^2}$

Analysing the Events Further

Edge involvements

- Each edge appears in $n-2$ events E_I and $\binom{n-2}{\mathrm{k}-2}$ events F_J

Dependencies

- $\Rightarrow$ each E_I depends on fewer than $d_1 := 3\left(n-2+\binom{n-2}{k-2}\right)$ other events
- $\Rightarrow$ each F_J depends on fewer than $d_2 := \binom{k}{2}\left(\mathrm{n}-2+\binom{\mathrm{n}-2}{\mathrm{k}-2}\right)$ events
- $\Rightarrow$ need to take $\mathrm{d} = \max\{d_1, d_2\} = d_2$ in the Local Lemma

Bounding n

- Thus $ep'(d+1) \leq e(p')^3 d \sim e\left(\frac{12\ln k}{k^2}\right)^3 \binom{k}{2}\left(n-2+\binom{n-2}{\mathrm{k}-2}\right)$

$$\leq \frac{12^3 e \ln^3 k}{k^4}\binom{n-2}{k-2} \leq \frac{12^3 e \ln^3 k}{n^2 k^2}\binom{n}{k} \leq \frac{12^3 e \ln^3 k}{n^2 k^2}\left(\frac{ne}{k}\right)^k$$

- For this to be less than 1, need $n = O(k)$

Post Mortem

Different kinds of events

- Triangle events E_I:
  - Probability $p_1 = p^3$
  - Depend on relatively few other events
- Independent set events F_J:
  - Probability $p_2 = (1-p)^{\binom{k}{2}}$
  - Depend on many other events

A possible remedy

- Wasteful to use same probability, dependency bounds for all events
- Triangle events are “more independent”
  - Could afford to let them occur with higher probability
- Ideally – track each event’s individual probability and dependencies

Tracking Dependencies

Representing dependence

- Keep track of dependencies using a directed graph
- Events are independent of their non-neighbours

Definition 4.2.1 (Dependency digraph)

Given events $E_1, E_2, \dots, E_m$, a directed graph $\vec{D}$ on the vertices $[m]$ is a *dependency digraph* if, for each $i \in [m]$, the event E_i is mutually independent of the set of events $\{E_j : (i,j) \notin \vec{D}\}$.

Why a digraph?

- In most applications, digraph will be symmetric
  - $(i,j) \in \vec{D} \Leftrightarrow (j,i) \in \vec{D}$
- Can sometimes help to have flexibility

The Lovász Local Lemma

Theorem 4.2.2 (Lovász Local Lemma; Erdős-Lovász, 1975)

Let $E_1, E_2, \ldots, E_m$ be events with a dependency digraph $\vec{D}$. If there are $x_i \in [0,1)$ such that $\mathbb{P}(E_i) \leq x_i \prod_{(i,j)\in\vec{D}} (1 - x_j)$ for all $i \in [m]$, then $\mathbb{P}(\cap_i E_i^c) \geq \prod_i (1 - x_i)$.

Special case: independent events

- Can take $\vec{D}$ to be edge-less
- $\Rightarrow$ suffices to have $x_i = \mathbb{P}(E_i)$, and done

General case

- Dependencies $\rightarrow$ correction factor $\prod_{(i,j)} (1 - x_j)$
- The more dependencies, the smaller this factor
- $\Rightarrow$ need probability of these events to shrink

Returning to $R(3,k)$ Once Again

Symmetries

- All triangle events E_I have the same probability and dependencies
  - $\Rightarrow$ should set $x_I = x$ for some common x
- All independent set events F_J also share the same parameters
  - $\Rightarrow$ set $x_J = y$ for some common y

Probability conditions

- Triangle events
  - Depend on at most $3(n-2) < 3n$ other triangle events
  - Depend on at most $\binom{n}{k}$ independent set bounds
  - $\Rightarrow$ suffices to have $\mathbb{P}(E_I) = p^3 \leq x(1-x)^{3n}(1-y)^{\binom{n}{k}}$
- Independent set events
  - Depend on at most $\binom{k}{2}(n-2) < \binom{k}{2}n$ triangle events, $\binom{n}{k}$ independent set events
  - $\Rightarrow$ suffices to have $\mathbb{P}(F_J) = (1-p)^{\binom{k}{2}} \leq y(1-x)^{\binom{k}{2}n}(1-y)^{\binom{n}{k}}$

A Conditional Result

Theorem 4.2.3 (Spencer, 1977)

Let $k, n \in \mathbb{N}$. If there are $p, x, y \in [0,1)$ such that

$$p^3 \leq x(1-x)^{3n}(1-y)^{\binom{n}{k}}$$

and

$$(1-p)^{\binom{k}{2}} \leq y(1-x)^{\binom{k}{2}n}(1-y)^{\binom{n}{\mathrm{k}}},$$

then $R(3,k) > n$.

Proof

- Follows immediately from Lovász Local Lemma and previous calculations ■

Optimisation

- Want to maximise n while satisfying the two inequalities

Some Heuristics

Maximise n subject to

(1) $\quad p^3 \leq x(1-x)^{3n}(1-y)^{\binom{n}{k}}$

(2) $\quad (1-p)^{\binom{k}{2}} \leq y(1-x)^{\binom{k}{2}n}(1-y)^{\binom{n}{k}}$

Setting y

- Do not want $(1-y)^{\binom{n}{k}} \approx e^{-y\binom{n}{k}}$ to be exponentially small
- $\Rightarrow$ set $y = \binom{n}{k}^{-1} \Rightarrow (1-y)^{\binom{n}{k}}$ is constant

Understanding x

- From (2), we need $(1-x)^n > 1-p$
- $\Rightarrow nx < p$, and $(1-x)^n$ is constant

More Heuristics

Maximise n subject to

(1’) $p^3 \leq x$

(2’) $(1-p)^{\binom{k}{2}} \leq y$

(3) $y = \binom{n}{k}^{-1}$ and $nx < p$

Setting p

- From (1’) and (3), $np^3 \leq nx < p \Rightarrow p < n^{-1/2}$

Fixing n

- From (2’) and (3), $(1-p)^{\binom{k}{2}} \approx e^{-p\binom{k}{2}} \leq \binom{n}{k}^{-1} \approx \left(\frac{k}{n}\right)^k$
- $\Rightarrow e^{-pk/2} \leq \frac{k}{n} \leq k^{-3/2}$, since $n = O(k^2)$
- $\Rightarrow k \geq p^{-1} \ln k = n^{1/2} \ln k \Rightarrow n = O\left(\left(\frac{k}{\ln k}\right)^2\right)$

Wrapping Things Up Neatly

Corollary 4.2.4

As $k \to \infty$, we have

$$\Omega\left(\left(\frac{k}{\ln k}\right)^2\right) = R(3, k) = O(k^2).$$

Proof

- Upper bound from Erdős-Szekeres
- Lower bound:
  - Choose $k \geq 20\sqrt{n}\ln n$, $y = \binom{n}{k}^{-1}$, $x = \frac{1}{9n^{-3/2}}$ and $p = \frac{1}{3\sqrt{n}}$
  - Substitute values into Theorem 4.2.3

■

Any questions?

§3 Proving the Local Lemma

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The Local Lemmas

- Recall the symmetric version

Theorem 4.1.1 (Symmetric Lovász Local Lemma; Erdős-Lovász, 1975)

Let $E_1, E_2, \ldots, E_m$ be events such that each event E_i is mutually independent of all but at most d of the other events, and $\mathbb{P}(E_i) \leq p$ for all i. If $ep(d+1) \leq 1$, then $\mathbb{P}(\cap_i E_i^c) > 0$.

- This follows easily from the general statement

Theorem 4.2.2 (Lovász Local Lemma; Erdős-Lovász, 1975)

Let $E_1, E_2, \ldots, E_m$ be events with a dependency digraph $\vec{D}$. If there are $x_i \in [0,1)$ such that $\mathbb{P}(E_i) \leq x_i \prod_{(i,j)\in\vec{D}}(1-x_j)$ for all $i \in [m]$, then $\mathbb{P}(\cap_i E_i^c) \geq \prod_i (1-x_i)$.

Deducing the Symmetric Statement

Theorem 4.1.1 (Symmetric Lovász Local Lemma; Erdős-Lovász, 1975)

Let $E_1, E_2, \ldots, E_m$ be events such that each event E_i is mutually independent of all but at most d of the other events, and $\mathbb{P}(E_i) \leq p$ for all i. If $ep(d+1) \leq 1$, then $\mathbb{P}(\cap_i E_i^c) > 0$.

Proof

- For each event E_i, set $x_i = \frac{1}{d+1}$
- Then $x_i \prod_{(i,j)\in\vec{D}}(1-x_j) = \frac{1}{d+1}\prod_{(i,j)\in\vec{D}}\left(1-\frac{1}{d+1}\right) \geq \frac{1}{d+1}\left(1-\frac{1}{d+1}\right)^d$
- For all $d \geq 1$, $\left(1-\frac{1}{d+1}\right)^d \geq e^{-1}$
- $\Rightarrow x_i \prod_{(i,j)\in\vec{D}}(1-x_j) \geq \frac{1}{e(d+1)} \geq p$
- $\Rightarrow \mathbb{P}(E_i) \leq p \leq x_i \prod_{(i,j)\in\vec{D}}(1-x_j) \Rightarrow \mathbb{P}(\cap_i E_i) \geq \left(1-\frac{1}{d+1}\right)^m > 0$ ■

Proving the General Statement

Theorem 4.2.2 (Lovász Local Lemma; Erdős-Lovász, 1975)

Let $E_1, E_2, \dots, E_m$ be events with a dependency digraph $\vec{D}$. If there are $x_i \in [0,1)$ such that $\mathbb{P}(E_i) \leq x_i \prod_{(i,j)\in\vec{D}}(1 - x_j)$ for all $i \in [m]$, then $\mathbb{P}(\cap_i E_i^c) \geq \prod_i(1 - x_i)$.

Chain rule

- $\mathbb{P}(\cap_{i=1}^m E_i^c) = \prod_{i=1}^m \mathbb{P}\left(E_i^c \middle| \cap_{j=1}^{i-1} E_j^c\right)$
  $= \prod_{i=1}^m \left(1 - \mathbb{P}\left(E_i \middle| \cap_{j=1}^{i-1} E_j^c\right)\right)$

New objective

- Suffices to show $\mathbb{P}\left(E_i^c \middle| \cap_{j=1}^{i-1} E_j^c\right) \geq 1 - x_i$ for each $i \in [m]$

The Irrelevance of Order

Objective

- For each $i \in [m]$, $\mathbb{P}\left(E_i^c \middle| \cap_{j=1}^{i-1} E_j^c\right) \geq 1 - x_i$

Reordering events

- If we reorder the events, the conditions do not change
- $\Rightarrow$ the event E_i could be preceded by any subset of the other events
- $\Rightarrow$ we can hope that more is true

Newer objective

- For each $i \in [m]$ and $S \subseteq [m] \setminus \{i\}$, $\mathbb{P}\left(E_i^c \middle| \cap_{j \in S} E_j^c\right) \geq 1 - x_i$

Conditional Probabilities

Objective

- For each $i \in [m]$ and $S \subseteq [m] \setminus \{i\}$, $\mathbb{P}\left(E_i^c \middle| \cap_{j \in S} E_j^c\right) \geq 1 - x_i$

Independence

- We know E_i is independent of some of the E_j
  - Conditioning on these events should be irrelevant
- Partition events
  - Let $S_1 = \left\{j \in S : (i,j) \in \vec{D}\right\}$
  - Let $S_2 = \left\{j \in S : (i,j) \notin \vec{D}\right\}$

Rewriting the probability

- $\mathbb{P}\left(E_i^c \middle| \cap_{j \in S} E_j^c\right) = 1 - \mathbb{P}\left(E_i \middle| \cap_{j \in S} E_j^c\right) = 1 - \frac{\mathbb{P}\left(E_i \cap \left(\cap_{\ell \in S_1} E_\ell^c\right) \middle| \left(\cap_{j \in S_2} E_j^c\right)\right)}{\mathbb{P}\left(\cap_{\ell \in S_1} E_\ell^c \middle| \cap_{j \in S_2} E_j^c\right)}$

Simplifying the Numerator

Recall

- $S_1 = \{j \in S: (i,j) \in \vec{D}\}$
- $S_2 = \{j \in S: (i,j) \notin \vec{D}\}$
- $\mathbb{P}(E_i | \cap_{j \in S} E_j^c) = \frac{\mathbb{P}\left(E_i \cap \left(\cap_{\ell \in S_1} E_\ell^c\right) \middle| \left(\cap_{j \in S_2} E_j^c\right)\right)}{\mathbb{P}\left(\cap_{\ell \in S_1} E_\ell^c \middle| \cap_{j \in S_2} E_j^c\right)}$

Numerator

- $E_i \cap \left(\cap_{\ell \in S_1} E_\ell^c\right) \subseteq E_i$
  - $\Rightarrow \mathbb{P}\left(E_i \cap \left(\cap_{\ell \in S_1} E_\ell^c\right) | \cap_{j \in S_2} E_j^c\right) \leq \mathbb{P}\left(E_i \middle| \cap_{j \in S_2} E_j^c\right)$
- E_i is mutually independent of the events in S_2
  - $\Rightarrow \mathbb{P}\left(E_i \middle| \cap_{j \in S_2} E_j^c\right) = \mathbb{P}(E_i)$
- Assumption: $\mathbb{P}(E_i) \leq x_i \prod_{j:(i,j) \in \vec{D}} (1 - x_j)$

Simplifying the Denominator

Objective

- For each $i \in [m]$ and $S \subseteq [m] \setminus \{i\}$, $\mathbb{P}\left(E_i^c \middle| \cap_{j \in S} E_j^c\right) \geq 1 - x_i$

Denominator: $\mathbb{P}\left(\cap_{\ell \in S_1} E_\ell^c \middle| \cap_{j \in S_2} E_j^c\right)$

- Chain rule
  - $\mathbb{P}\left(\cap_{\ell \in S_1} E_\ell^c \middle| \cap_{j \in S_2} E_j^c\right) = \prod_{\ell \in S_1} \mathbb{P}\left(E_\ell^c \middle| \left(\cap_{r \in S_1, r < \ell} E_r^c\right) \cap \left(\cap_{j \in S_2} E_j^c\right)\right)$
  - Let $T_\ell = \{r \in S_1 : r < \ell\} \cup S_2$
- Apply the objective
  - $\Rightarrow \mathbb{P}\left(E_\ell^c \middle| \cap_{j \in T_\ell} E_j^c\right) \geq 1 - x_\ell$
- Substitute in
  - $\Rightarrow \mathbb{P}\left(\cap_{\ell \in S_1} E_\ell^c \middle| \cap_{j \in S_2} E_j^c\right) \geq \prod_{\ell \in S_1} (1 - x_\ell)$
- $S_1 \subseteq \left\{j : (i, j) \in \vec{D}\right\}$
  - $\Rightarrow \mathbb{P}\left(\cap_{\ell \in S_1} E_\ell^c \middle| \cap_{j \in S_2} E_j^c\right) \geq \prod_{j : (i,j) \in \vec{D}} (1 - x_j)$

Achieving Our Objective

Objective

- For each $i \in [m]$ and $S \subseteq [m] \setminus \{i\}$, $\mathbb{P}\left(E_i^c \middle| \cap_{j \in S} E_j^c\right) \geq 1 - x_i$

Recall

- $\mathbb{P}\left(E_i^c \middle| \cap_{j \in S} E_j^c\right) = 1 - \dfrac{\mathbb{P}\left(E_i \cap \left(\cap_{\ell \in S_1} E_\ell^c\right) \middle| \cap_{j \in S_2} E_j^c\right)}{\mathbb{P}\left(\cap_{\ell \in S_1} E_\ell^c \middle| \cap_{j \in S_2} E_j^c\right)}$
- $\mathbb{P}\left(E_i \cap \left(\cap_{\ell \in S_1} E_\ell^c\right) \middle| \cap_{j \in S_2} E_j^c\right) \leq \mathbb{P}(E_i) \leq x_i \prod_{j:(i,j) \in \vec{D}} (1 - x_j)$
- $\mathbb{P}\left(\cap_{\ell \in S_1} E_\ell^c \middle| \cap_{j \in S_2} E_j^c\right) \geq \prod_{j:(i,j) \in \vec{D}} (1 - x_j)$

- $\Rightarrow \mathbb{P}\left(E_i^c \middle| \cap_{j \in S} E_j^c\right) \geq 1 - x_i$ $\blacksquare$

Circular logic

- We used the objective to lower bound the denominator

Induction to the Rescue

Objective

- For each $i \in [m]$ and $S \subseteq [m] \setminus \{i\}$, $\mathbb{P}\left(E_i^c \middle| \cap_{j \in S} E_j^c\right) \geq 1 - x_i$

The issue

- Used the objective when bounding the denominator in the proof
- Parameters
  - $i = \ell \in S_1$
  - $S = T_\ell = \{r \in S_1 : r < \ell\} \cup S_2$

The fix

- Size of conditioned set
  - We have $|T_\ell| \leq |S_1| - 1 + |S_2| < |S|$
  - $\Rightarrow$ when proving the objective for a set S, only require it for smaller subsets
- Apply induction on $|S|$
  - Base case: $S = \emptyset$ is trivial

Proof Recap

Theorem 4.2.2 (Lovász Local Lemma; Erdős-Lovász, 1975)

Let $E_1, E_2, \ldots, E_m$ be events with a dependency digraph $\vec{D}$. If there are $x_i \in [0,1)$ such that $\mathbb{P}(E_i) \leq x_i \prod_{(i,j)\in\vec{D}} (1 - x_j)$ for all $i \in [m]$, then $\mathbb{P}(\cap_i E_i^c) \geq \prod_i (1 - x_i)$.

Ideas

- Chain rule: probability of intersection is product of conditional probabilities
- Prove $\mathbb{P}\left(E_i^c \middle| \cap_{j\in S} E_j^c\right) \geq 1 - x_i$ by induction on $|S|$
    - Separate conditioned events by dependence of E_i
    - Simplify resulting expression by bounding the numerator
    - Apply induction hypothesis to the denominator
- Substituting into chain rule gives result

Any questions?

§4 Latin Transversals

Chapter 4: The Lovász Local Lemma

The Probabilistic Method

Latin Squares

Definition 4.4.1 (Latin square)

A *Latin square of order* n is an $n \times n$ array with entries from $[n]$ such that each symbol appears exactly once in each row and column.

1	**2**	**3**	**4**	**5**
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

Applications

- Experimental design
- Tournament scheduling
- Games and recreation
- Algebra
  - Cayley table of a group is a Latin square

Latin Transversals

Definition 4.4.2 (Latin transversal)

Given an $m \times n$ array with entries in $\mathbb{N}$, a *transversal* is a selection of cells without any repeated row, column or symbol.

1	**2**	**3**	**4**	**5**
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

Example

- Conducting a survey
- Public divided by metrics
  - Age
  - Height
  - No. combinatorics courses taken
- Want a fair sample
  - No group is overrepresented

Latin Transversals

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An Extremal Problem

Large transversals?

- How large a transversal must a Latin square of order n contain?

Proposition 4.4.3

Any Latin square of order n contains a transversal of size at least $\frac{n}{3}$.

Proof

- Build a transversal greedily
- There are a total of n^2 cells
- Each cell clashes with $3(n-1)$ other cells
  - Those in the same row, column or with the same symbol
- $\Rightarrow$ we can select at least $\frac{n^2}{3(n-1)+1}$ cells before we run out ■

An Upper Bound to the Extremal Problem

Proposition 4.4.4

For every even $n \in \mathbb{N}$, there is a Latin square of order n without a transversal of size n.

Proof

- Let $n = 2k$ and L be the Cayley table of $\mathbb{Z}_{2k}$
  - $L(i,j) \coloneqq i + j \pmod{2k}$
- Suppose we have a transversal
  - Chosen cells: $\{(i, \pi(i)) : i \in [2k]\}$ for some permutation $\pi \in S_{2k}$
- Rows, columns and symbols range over $[2k]$
  - $\Rightarrow$ sum is $\binom{2k}{2} = k(2k-1) \equiv k \pmod{2k}$
- But summing symbols = summing rows and columns:
  - $k \equiv \sum_i L(i, \pi(i)) = \sum_i (i + \pi(i)) = \sum_i i + \sum_i \pi(i) \equiv k + k \equiv 0 \not\equiv k \pmod{2k}$ ■

A Daring Conjecture

Conjecture 4.4.5 (Ryser-Brualdi-Stein, 1967+)

Every Latin square of order n admits a transversal of size $n - 1$.

Odd orders

- Ryser conjectured that odd Latin squares have full transversals

Theorem 4.4.6 (Erdős-Spencer, 1991)

Let A be an $n \times n$ array with entries in $\mathbb{N}$. If no symbol appears more than $\frac{n-1}{4e}$ times in A, then A admits a transversal of size n.

Comparison to the conjecture

- Weak: Latin squares have each symbol appearing n times
- Strong: In theorem, no restrictions on row or column repetitions!

Proof Framework

Goal

- Show that a random permutation can give a full transversal

Probability space

- Choose $\pi \in S_n$ uniformly at random
- Potential transversal $\{(i, \pi(i)) : i \in [n]\}$

Bad events

- Chosen cells from distinct rows and columns by construction
- Only way to fail: repeat a symbol
- How do we define events to capture this?

Defining Failure

Events by symbol

- For each symbol $i \in \mathbb{N}$ appearing in A, define an event
  - $E_i = \{\text{two cells with the symbol } i \text{ are selected}\}$
- $\{\pi \text{ gives a transversal}\} = \cap_i E_i^c$

Probabilities

- $\mathbb{P}(E_i)$ depends very much on structure of A
  - If all i-entries are in the same row $\Rightarrow \mathbb{P}(E_i) = 0$
  - If $\Omega(n)$ are on a diagonal $\Rightarrow \mathbb{P}(E_i) = \Omega(1)$
- Expected number of events
  - Hard to compute
  - Could grow linearly

Many dependencies

- Different symbols that appear in the same row/column are dependent

Redefining Failure

Events by rows

- For $\{i,j\} \in \binom{[n]}{2}$, define $E_{i,j} = \{A_{i,\pi(i)} = A_{j,\pi(j)}\}$
- $\{\pi \text{ gives a transversal}\} = \cap_{\{i,j\}} E_{i,j}^c$

Same issues as before

- Probabilities depend heavily on array A
- Expected number of events can be $\Omega(n)$
- High dependence
  - Knowing $E_{i,j}$ occurs could tell us which elements are selected
  - Affects distribution in other rows

Reredefining Failure

Events by cells

- Identify exactly where symbols are repeated
- For every pair of cells (i, j) and (i', j') with
  - $A(i, j) = A(i', j')$
  - $i \neq i'$
  - $j \neq j'$

  define the event $E_{i,j,i',j'} = \{\pi(i) = j\} \cap \{\pi(i') = j'\}$
- $\{\pi \text{ gives a transversal}\} = \cap_{i,j,i',j'} E_{i,j,i',j'}$

Probabilities

- Each event occurs with probability $\frac{1}{n(n-1)}$
- Number of events depends on structure of A, but at most $\frac{1}{2} \cdot n^2 \cdot \frac{n-1}{4e}$
- $\Rightarrow$ expected number of bad events can still be $\Omega(n)$

Examining Independence

Neighbouring events

- Consider events $E_{i,j,i',j'}$ and $E_{p,q,p',q'}$
  - Correspond to cells (i, j), (i', j'), (p, q) and (p', q')
- Only one cell selected from each row/column
- $\Rightarrow$ if $\{i, i'\} \cap \{p, p'\} \neq \emptyset$ or $\{j, j'\} \cap \{q, q'\} \neq \emptyset$, no independence

Non-neighbouring events

- Permutation restrictions are global in nature
- $\Rightarrow$ information travels even when not sharing a row or column
- $\Rightarrow$ cannot expect any independence

Who Needs Independence?

Proof of Lovász Local Lemma

- Used independence when bounding the numerator
- $\mathbb{P}\left(E_i \middle| \cap_{j \in S_2} E_j^c\right) = \mathbb{P}(E_i) \leq x_i \prod_{j:(i,j) \in \vec{D}} \left(1 - x_j\right)$
  - (First) equality by independence
  - (Second) inequality by assumption

Weakening condition

- We only use the inequality
  - Could skip the intermediate equality
- $\Rightarrow$ suffices to have $\mathbb{P}\left(E_i \middle| \cap_{j \in S} E_j^c\right) \leq x_i \prod_{j:(i,j) \in \vec{D}} \left(1 - x_j\right)$ for all $i \in [m]$ and $S \subseteq [m] \setminus \left\{j : (i,j) \in \vec{D}\right\}$

The Lopsided Lovász Local Lemma

Strengthened result

- Using this observation, we can weaken the requirement in the Local Lemma
- Following version is useful in spaces with limited independence
  - Most pairs of events should be positively correlated

Theorem 4.4.7 (Lopsided Lovász Local Lemma; Erdős-Spencer, 1991)

Let $E_1, E_2, \dots, E_m$ be events in a probability space, let $x_1, x_2, \dots, x_m \in [0,1)$, and let $\vec{D}$ be a directed graph on the vertices $[m]$. If, for every $i \in [m]$ and $S \subseteq [m] \setminus \left(\{j : (i,j) \in \vec{D}\} \cup \{i\}\right)$, we have

$$\mathbb{P}\left(E_i \middle| \cap_{j \in S} E_j^c\right) \leq x_i \prod_{j:(i,j) \in \vec{D}} (1 - x_j),$$

then $\mathbb{P}(\cap_i E_i^c) \geq \prod_i (1 - x_i) > 0.$

Verifying the Condition

Lemma 4.4.8

Let the events $E_{i,j,i',j'}$ be as previously defined, and let S be a set of indices for events involving cells not sharing a row or column with (i, j) or (i', j'). Then

$$\mathbb{P}\left(E_{i,j,i',j'} \middle| \cap_{(p,q,p',q')\in S} E^{c}_{p,q,p',q'}\right) \leq \frac{1}{n(n-1)}.$$

Proof idea

- Without loss of generality, may assume $i = j = 1$, $i' = j' = 2$
- Restrict to permutations π satisfying $\cap_{(p,q,p',q')\in S} E^{c}_{p,q,p',q'}$
- By modifying permutations, show that number of permutations with $\pi(1) = r$ and $\pi(2) = s$ is minimised (for $r \neq s$) when $r = 1$ and $s = 2$

Verifying the Condition - Notation

Objective

- $\mathbb{P}\left(E_{1,1,2,2} \,\middle|\, \cap_{(p,q,p',q')\in S} E^c_{p,q,p',q'}\right) \leq \frac{1}{n(n-1)}$

Notation

- Call π "good" if $\pi \in \cap_{(p,q,p',q')\in S} E^c_{p,q,p',q'}$
- Let $P_{r,s} = \{\pi \text{ good},\ \pi(1) = r,\ \pi(2) = s\}$
- Goal: $|P_{1,2}| \leq |P_{r,s}|$ for all $(r,s) \in [n]^2, r \neq s$

Setting up the proof

- Goal: Construct an injection $P_{1,2} \hookrightarrow P_{r,s}$
- Case: $r, s \notin \{1,2\}$ (others similar)
- Let $\pi \in P_{1,2}$ and let $x = \pi^{-1}(r), y = \pi^{-1}(s)$

Verifying the Condition - Proof

Goal

- Injection $P_{1,2} \hookrightarrow P_{r,s}$
- Given: $\pi \in P_{1,2}, \pi(x) = r, \pi(y) = s$

Switching

- Define new permutation $\pi^* \in P_{r,s}$
- $\pi^*(z) = \begin{cases} r \text{ if } z = 1 \\ s \text{ if } z = 2 \\ 1 \text{ if } z = x \\ 2 \text{ if } z = y \\ \pi(z) \text{ otherwise} \end{cases}$
- π^* is good: only change cells in the first two rows or columns, avoiding S
- $\Rightarrow \pi^* \in P_{r,s}$
- The map $\pi \mapsto \pi^*$ is injective ∎

Finding Large Transversals

Theorem 4.4.6 (Erdős-Spencer, 1991)

Let A be an $n \times n$ array with entries in $\mathbb{N}$. If no symbol appears more than $\frac{n-1}{4e}$ times in A, then A admits a transversal of size n.

Proof

- We will apply the Lopsided Lovász Local Lemma
- $\vec{D}$: edges between (i, j, i', j') and (p, q, p', q') if the corresponding cells share a row or column
  - Each event is adjacent to at most $\mathrm{d} \coloneqq 4 \cdot n \cdot \frac{n-1}{4e} - 1 = \frac{\mathrm{n(n-1)}}{\mathrm{e}} - 1$ other events
- We set $x_{i,j,i',j'} = \frac{1}{d+1}$ for each event
  - $\Rightarrow$ inequality reduces to $ep(d+1) \leq 1$, as in symmetric case
- $\mathbb{P}\left(E_{i,j,i',j'}\right) = \frac{1}{n(n-1)} = \frac{1}{\mathrm{e(d+1)}}$. ∎

Ryser's Conjecture

State of the art

- More involved probabilistic proofs bring us much closer to Ryser's Conjecture

Theorem (Keevash-Pokrovskiy-Sudakov-Yepremyan, 2020+)

Every Latin square of order n admits a transversal of size

$$n - O\left(\frac{\log n}{\log\log n}\right).$$

Theorem (Kwan, 2016+)

Almost all Latin squares of order n have a transversal of size n.

Any questions?