Chapter 5: Concentration

The Probabilistic Method

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Chapter Overview

- Prove some strong concentration inequalities
- Improve bounds on Ramsey numbers
- Study Hamiltonicity and chromatic number of $G(n, p)$

§1 Chernoff Bounds

Chapter 5: Concentration

The Probabilistic Method

Domination vs Minimum Degree

Corollary 2.2.5

Let G be an n-vertex graph with $\delta(G) \geq \delta$. Then G has a dominating set $S \subseteq V(G)$ with $|S| \leq \left(\frac{\ln(\delta+1)+1}{\delta+1}\right) n$.

Homework exercise

- Show bound is tight by consider $G(n,p)$

Degrees in $G(n,p)$

- Degree of a vertex $\sim \mathrm{Bin}(n-1,p)$
- $\Rightarrow$ expected degree is $(n-1)p$
- Minimum degree: need to show not far from mean
  - Suppose $\mathbb{P}(|\deg(v) - (n-1)p| \geq a) < \frac{1}{2n}$
  - Union bound $\Rightarrow \mathbb{P}\big(\delta\big(G(n,p)\big) \geq (n-1)p - d\big) > \frac{1}{2}$

Comparing Bounds

Concentration inequalities

- Let $X \sim \text{Bin}(n-1, p), p \leq \frac{1}{2}$
  - $\mathbb{E}[X] = (n-1)p$, $\text{Var}(X) = (n-1)p(1-p) = \Theta(np)$
- Markov: $\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$
  - $\Rightarrow$ error probability $< \frac{1}{2n}$ for $a = \Omega(n)$
- Chebyshev: $\mathbb{P}(|X - \mathbb{E}[X]| \geq a) \leq \frac{\text{Var}(X)}{a^2}$
  - $\Rightarrow$ error probability $< \frac{1}{2n}$ for $a = \Omega(n\sqrt{p})$
- Central Limit Theorem: $\mathbb{P}(|X - \mathbb{E}[X]| \geq a) \leq \exp\left(\frac{-ca^2}{\text{Var}(X)}\right)$
  - $\Rightarrow$ error probability $< \frac{1}{2n}$ for $a = \Omega(\sqrt{np}\log n)$

The Problem With CLT

Asymptotics

- Central Limit Theorem is asymptotic, valid as $n \to \infty$
- We would like a quantitative bound for some given n

Definition 5.1.1

Let $S_n = \sum_{i=1}^{n} X_i$, where the X_i are independently and uniformly distributed on $\{-1,1\}$.

Binomial connection

- We have $S_n \sim 2\,\mathrm{Bin}\left(n, \frac{1}{2}\right) - \frac{n}{2}$
  - Convenient to translate so mean is zero

Goal

- Show S_n is exponentially unlikely to be far from zero

Chernoff Bounds

Theorem 5.1.2 (Symmetric Chernoff Bound)

For every $a > 0$, we have $\mathbb{P}(S_n \geq a) \leq \exp\left(-\frac{a^2}{2n}\right)$.

Remarks

- Concrete bounds for all n, a
- Symmetry: same bound for $\mathbb{P}(S_n \leq -a)$
- $\mathrm{Bin}\left(n, \frac{1}{2}\right) = \frac{1}{2}(S_n + n)$
  - $\Rightarrow$ concentration for binomial random variables

Corollary 5.1.3

For every $a > 0$, we have $\mathbb{P}\left(\left|\mathrm{Bin}\left(n, \frac{1}{2}\right) - \frac{n}{2}\right| \geq a\right) \leq 2\exp\left(\frac{-2a^2}{n}\right)$.

Proving Chernoff

Theorem 5.1.2 (Symmetric Chernoff Bound)

For every $a > 0$, we have $\mathbb{P}(S_n \geq a) \leq \exp\left(-\frac{a^2}{2n}\right)$.

Proof

- Exponential conversion
  - $\{S_n \geq a\} = \{e^{S_n} \geq e^a\} = \{e^{\lambda S_n} \geq e^{\lambda a}\}$
- Concentration
  - $e^{\lambda S_n}$ a non-negative random variable
  - Markov: $\mathbb{P}\left(e^{\lambda S_n} \geq e^{\lambda a}\right) \leq \mathbb{E}\left[e^{\lambda S_n}\right] e^{-\lambda a}$
- Expectation
  - Recall $S_n = \sum_{i=1}^n X_i$
  - $\Rightarrow e^{\lambda S_n} = \prod_{i=1}^n e^{\lambda X_i}$
  - Independence $\Rightarrow \mathbb{E}\left[e^{\lambda S_n}\right] = \prod_{i=1}^n \mathbb{E}\left[e^{\lambda X_i}\right] = \left(\frac{1}{2}\left(e^{\lambda} + e^{-\lambda}\right)\right)^n = \cosh^n(\lambda)$

Over the Cosh

Recall

- $\mathbb{P}(S_n \geq a) \leq \mathbb{E}\left[e^{\lambda S_n}\right]e^{-\lambda a}$
- $\mathbb{E}\left[e^{\lambda S_n}\right] = \cosh^n(\lambda)$

A little calculus

- $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$
- Taylor series: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$
- $\Rightarrow \cosh(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots \leq 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \cdots = e^{\frac{x^2}{2}}$

Finishing the proof

- $\therefore \mathbb{P}(S_n \geq a) \leq \exp\left(\frac{1}{2}n\lambda^2 - \lambda a\right)$
- Minimise: $\lambda = \frac{a}{n} \Rightarrow \mathbb{P}(S_n \geq a) \leq \exp\left(-\frac{a^2}{2n}\right)$ ∎

The General Setting

Shortcomings

- Required each X_i to be uniform on $\{-1,1\}$

Wider Framework

- $p_1, p_2, \ldots, p_n \in [0,1]$, and $p = n^{-1} \sum_{i=1}^{n} p_i$
- X_i independent with $\mathbb{P}(X_i = 1 - p_i) = p_i$ and $\mathbb{P}(X = -p_i) = 1 - p_i$
- $X = \sum_{i=1}^{n} X_i$

Theorem 5.1.4 (Asymmetric Chernoff Bound)

Let $a > 0$ and let X and p be as above. Then

$$\mathbb{P}(X \leq -a) \leq \exp\left(-\frac{a^2}{2pn}\right) \text{ and } \mathbb{P}(X \geq a) \leq \exp\left(\frac{-a^2}{2pn} + \frac{a^3}{2(pn)^2}\right).$$

An Asymmetric Chernoff Bound

Theorem 5.1.4 (Asymmetric Chernoff Bound)

Let $a > 0$ and let X and p be as above. Then

$$\mathbb{P}(X \leq -a) \leq \exp\left(\frac{-a^2}{2pn}\right) \text{ and } \mathbb{P}(X \geq a) \leq \exp\left(\frac{-a^2}{2pn} + \frac{a^3}{2(pn)^2}\right).$$

Special case

- $p_i = p$ for all $i \Rightarrow X + np \sim \mathrm{Bin}(n, p)$
- $\Rightarrow \mathbb{P}(|\mathrm{Bin}(n,p) - np| \geq a) \leq 2\exp\left(\frac{-a^2}{2pn} + \frac{a^3}{2(pn)^2}\right)$

Corollary 5.1.5

For every $\varepsilon > 0$ there is some $c_\varepsilon > 0$ such that, if Y is the sum of mutually independent indicator random variables and $\mu = \mathbb{E}[Y]$, then $\mathbb{P}(|Y - \mu| \geq \varepsilon\mu) \leq 2\exp(-c_\varepsilon \mu)$.

Any questions?

§2 Returning to Ramsey

Chapter 5: Concentration

The Probabilistic Method

The Story So Far

Goal

- Determine the order of magnitude of $R(3,k)$

Upper bound

- Erdős-Szekeres (1935): $R(3,k) \le \binom{k+1}{2} = O(k^2)$

Lower bounds

- First moment, Mantel: $R(3,k) = \Omega(k)$
- Alterations: $R(3,k) = \Omega\left(\left(\frac{k}{\log k}\right)^{3/2}\right)$
- Lovász Local Lemma: $R(3,k) = \Omega\left(\left(\frac{k}{\log k}\right)^{2}\right)$

Alterations Revisited

Theorem 2.1.2 ($\ell = 3$)

For every $n, k \in \mathbb{N}$ and $p \in [0,1]$, we have

$$R(3,k) > n - \binom{n}{3} p^3 - \binom{n}{k} (1-p)^{\binom{k}{2}}.$$

Proof sketch

- Take $G \sim G(n,p)$
- Remove one vertex from each triangle, independent set of size k
  - Resulting graph G' is Ramsey
- First moment $\Rightarrow$ with positive probability G' has many vertices

Optimisation

- Largest right-hand side can be is $O\left(\left(\frac{k}{\log k}\right)^{3/2}\right)$

Alternative Alterations

Vertex removal

- Wasteful operation
  - To fix a single, small triangle, we make $\Omega(n)$ changes to the graph
  - Shrinks our resulting Ramsey graph too much

Edge removal

- More efficient fix
  - To fix a triangle, need only remove a single edge
- Problematic
  - Being triangle-free and having small independence numbers are in conflict
  - Need to ensure we can destroy all triangles without creating large independent sets
- A new hope
  - Can our more advanced probabilistic tools help?

Plan of Attack

Detriangulation

- Need to remove at least one edge from each triangle
- Let $\mathcal{T}$ be a maximal set of *edge-disjoint* triangles in G
  - If T is a triangle in G, maximality $\Rightarrow$ must share an edge with some $T' \in \mathcal{T}$
- Remove all edges of all triangles in $\mathcal{T}$
  - Removes $3|\mathcal{T}|$ edges
  - Need to remove at least $|\mathcal{T}|$ edges

Independent sets

- Cannot let a set S of k vertices become independent
- Would help if $G[S]$ had many edges to begin with
  - *Expect* to see $\binom{k}{2}p$ edges
  - Chernoff $\Rightarrow$ very unlikely to see many fewer
  - Can afford a union bound over all such sets S

Local Edge Distribution

Local edge counts

- Fix a set S of k vertices
- $e(G[S]) \sim \mathrm{Bin}\left(\binom{k}{2}, p\right)$
- Expect $\binom{k}{2}p$ edges, how likely are we to see at least half of that?

Theorem 5.1.4 (Asymmetric Chernoff Bound)

Let $a > 0$ and let X and p be as before. Then $\mathbb{P}(X \leq -a) \leq \exp\left(\frac{-a^2}{2pn}\right)$.

Applying Chernoff

- Set $X = \mathrm{Bin}\left(\binom{k}{2}, p\right) - \binom{k}{2}p$, and let $a = \frac{1}{2}\binom{k}{2}p$
- $\Rightarrow \mathbb{P}\left(e(G[S]) \leq \frac{1}{2}\binom{k}{2}p\right) \leq \exp\left(-\frac{1}{8}\binom{k}{2}p\right)$

Local Properties Globally

Recall

- $\mathbb{P}\left(e(G[S]) \leq \frac{1}{2}\binom{k}{2}p\right) \leq \exp\left(-\frac{1}{8}\binom{k}{2}p\right)$

Union bound

- We need *every* k-set to have many edges
  - Apply a union bound over choice of S
- $\mathbb{P}\left(\exists S\colon e(G[S]) \leq \frac{1}{2}\binom{k}{2}p\right) \leq \binom{n}{k}\exp\left(-\frac{1}{8}\binom{k}{2}p\right) \leq \exp\left(k\ln n - \frac{1}{8}\binom{k}{2}p\right)$

Setting parameters

- Small if $k\ln n \leq \frac{1}{10}\binom{k}{2}p$, say
- $\Leftrightarrow p \geq \frac{20\ln n}{k-1}$
  - To avoid too many triangles, take equality above
- Then with high probability each k-set spans at least $\frac{1}{2}\binom{k}{2}p = 5k\ln n$ edges

A Tangle of Triangles

Recall

- Setting $p = \frac{20 \ln n}{k-1} \Rightarrow$ almost surely, every k-set has at least $5k \ln n$ edges

New independent sets

- Remove all edges from a maximal set $\mathcal{T}$ of edge-disjoint triangles
- Need to avoid creating an independent set of k vertices
- Fix a set S of k vertices

How many edges do we lose?

- Only triangles with an edge in S are relevant
- Number of potential such triangles:
  - $\binom{k}{3} + \binom{k}{2}(n-k)$
- Expected number of relevant triangles
  - $\left(\binom{k}{3} + \binom{k}{2}(n-k)\right) p^3 \approx \frac{4000}{3} \ln^3 n + 4000 \frac{n-k}{k} \ln^3 n \approx 4000 \frac{n}{k} \ln^3 n$

Accounting for Triangles

Recall

- With high probability, each k-set S spans at least $5k \ln n$ edges
- Expect there to be at most $4000 \frac{n}{k} \ln^3 n$ triangles with an edge in S

Setting more parameters

- In order to ensure S does not become independent, need $\frac{n}{k} \ln^3 n \leq ck \ln n$
  - $c > 0$ some small constant
- Solving: $n \leq c \left(\frac{k}{\ln n}\right)^2 = c' \left(\frac{k}{\log k}\right)^2$

Large deviations

- Need to ensure that no set S sees too many triangles
  - Union bound over $\binom{n}{k}$ many sets
- $\Rightarrow$ need the probability that we get more triangles than expected to be small

Too Many Triangles

Concentration inequalities

- Chernoff: probability of seeing too many triangles is exponentially small
  - Problem: indicator variables for triangles are not independent
- Chebyshev: error probabilities only polynomially small
  - Not enough to make up for $\binom{n}{k}$ summands in union bound

Saving grace

- We only remove edges of triangles in $\mathcal{T}$, edge-disjoint set of triangles

Lemma 5.2.1 (Erdős-Tetali, 1990)

Let $E_1, E_2, \ldots, E_m$ be a collection of events and set $\mu = \sum_{i=1}^{m} \mathbb{P}(E_i)$. For any s,

$$\mathbb{P}\left(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_s} \text{ for some independent } E_{i_1}, E_{i_2}, \ldots, E_{i_s}\right) \leq \frac{\mu^s}{s!}.$$

The Erdős-Tetali Lemma

Lemma 5.2.1 (Erdős-Tetali, 1990)

Let $E_1, E_2, \ldots, E_m$ be a collection of events and set $\mu = \sum_{i=1}^{m} \mathbb{P}(E_i)$. For any s,

$$\mathbb{P}\left(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_s} \text{ for some independent } E_{i_1}, E_{i_2}, \ldots, E_{i_s}\right) \le \frac{\mu^s}{s!}.$$

Proof

- Take a union bound over all such s-sets of events
- $\mathbb{P}\left(E_{i_1} \cap \cdots \cap E_{i_s} \text{ for some independent events}\right)$

$$\le \sum_{\{i_1,\ldots,i_s\} \text{ ind}} \mathbb{P}\left(E_{i_1} \cap \cdots \cap E_{i_s}\right) = \frac{1}{s!} \sum_{(i_1,\ldots,i_s) \text{ ind}} \mathbb{P}\left(E_{i_1} \cap \cdots \cap E_{i_s}\right)$$

$$= \frac{1}{s!} \sum_{(i_1,\ldots,i_s) \text{ ind}} \prod_{j=1}^{s} \mathbb{P}\left(E_{i_j}\right) \quad \le \frac{1}{s!} \sum_{(i_1,\ldots,i_s) \in [m]^s} \prod_{j=1}^{s} \mathbb{P}\left(E_{i_j}\right)$$

$$= \frac{1}{s!} \left(\sum_{i \in [m]} \mathbb{P}(E_i)\right)^s \quad = \frac{\mu^s}{s!}$$

∎

Handling Triangle Errors

Recall

- With high probability, each k-set S has at least $5k \ln n$ edges
- Expected number of triangles with an edge in S at most $ck \ln n$ for small c

Erdős-Tetali

- Events E_i: ith triangle meeting S is present in $G(n, p)$
  - $\mu \leq ck \ln n$
- Let $s = k \ln n$
- Lemma 5.2.1 $\Rightarrow \mathbb{P}(S \text{ sees edges of } s \text{ disjoint triangles}) \leq \frac{\mu^s}{s!}$

Calculation

- Stirling: $s! \geq \left(\frac{s}{e}\right)^s$
- $\Rightarrow \frac{\mu^s}{s!} \leq \left(\frac{\mu e}{s}\right)^s \leq (ce)^{k \ln n} < n^{-k}$ if $c < e^{-2}$

Completing the Proof

Union bound

- Union bound over all $\binom{\mathrm{n}}{k} < n^k$ sets $\Rightarrow$ with high probability, every k-set:
  - Spans at least $5k \ln n$ edges and meets at most $k \ln n$ edge-disjoint triangles

Alteration

- Given $G \sim G(n, p)$, where $n = c' \left(\frac{k}{\log k}\right)^2$ and $p = \frac{20 \ln n}{k-1}$
- Let $\mathcal{T}$ be a maximal set of edge-disjoint triangles, and remove all edges in $\mathcal{T}$
  - Each k-set loses at most $3k \ln n$ edges $\Rightarrow$ doesn't become independent
- Resulting graph is therefore Ramsey. ■

Theorem 5.2.2 (Erdős, 1961; Krivelevich, 1995)

As $k \to \infty$, $R(3, k) = \Omega\left(\left(\frac{k}{\log k}\right)^2\right)$.

Closing In

Lower bounds

- Edge-alteration gave same bound as Lovász Local Lemma
  - $R(3,k) = \Omega\left(\left(\frac{k}{\log k}\right)^2\right)$
- Could this be the truth? What can we say in the other direction?

Theorem 1.5.5 (Erdős-Szekeres, 1935)

For all $\ell, k \in \mathbb{N}$,

$$R(\ell,k) \leq \binom{\ell+k-2}{\ell-1} = O\left(k^{\ell-1}\right).$$

In particular, $R(3,k) = O(k^2)$.

Narrowing the gap

- Left with a $\log^2 k$ gap to close

Independent Sets in Triangle-Free Graphs

Proposition 5.2.3

If G is an n-vertex triangle-free graph, $\alpha(G) \geq \sqrt{n} - 1$.

Proof

- Key observation: G triangle-free $\Rightarrow$ every neighbourhood is independent
- $\therefore$ if G has a vertex of degree $\sqrt{n} - 1$, we are done
    - Otherwise $\Delta(G) < \sqrt{n} - 1$
- Greedy algorithm:
    - $\alpha(G) \geq \frac{n}{\Delta(G)+1} \geq \sqrt{n}$ ∎

Ramsey numbers

- Implies $R(3, k) = O(k^2)$

Room for Improvement

Greedy algorithm

- Order vertices arbitrarily
- Add first vertex v to independent set
- Remove all its $\leq \Delta$ neighbours, and repeat
- Bound is sharp only if v never has any neighbours previously removed
    - Only true for disjoint union of cliques
    - $\Rightarrow$ cannot be sharp for triangle-free graphs

Theorem 5.2.4 (Ajtai, Komlós, Szemerédi, 1980; Shearer, 1995)

If G is an n-vertex triangle-free graph with maximum degree Δ, then

$$\alpha(G) \geq \frac{n \log \Delta}{8\Delta}.$$

An Improved Upper Bound

Corollary 5.2.5

As $k \to \infty$, $R(3,k) \le \dfrac{8k^2}{\log k}$.

Proof

- Let $n = \dfrac{8k^2}{\log k}$ and let G be an n-vertex triangle-free graph
- If $\Delta(G) \ge k$
  - Let v be a vertex of maximum degree
  - $N(v)$ is an independent set of size $\ge k$
- If $\Delta(G) < k$
  - Theorem 5.2.4 $\Rightarrow \alpha(G) \ge \dfrac{n \log \Delta}{8\Delta} \ge \dfrac{n \log k}{8k} = k$ ∎

The Big Picture

Randomness

- We show that a *random* independent set I of G has this size
- If we let $Y_v = 1_{\{v \in I\}}$, then $|I| = \sum_v Y_v$
  - Would suffice to compute $\mathbb{E}[|I|] = \sum_v \mathbb{E}[Y_v] = \sum_v \mathbb{P}(v \in I)$
  - Computing $\mathbb{P}(v \in I)$ not straightforward – depends on neighbourhood

Neighbourhoods

- How does I meet the neighbourhood $N(v)$?
- If $v \in I$:
  - Must have $I \cap N(v) = \emptyset$
- If $v \notin I$:
  - Can have $I \cap N(v) \neq \emptyset$
  - Since $N(v)$ is independent, intersection can be arbitrary
  - $\Rightarrow$ might expect intersection to be large

New Random Variables

Local variables

- Define new variables to account for local information
- Let $X_v = \Delta \cdot 1_{\{v \in I\}} + |I \cap N(v)|$
- Heuristic justification
  - Regularise contribution of v
    - When $v \in I$, have $X_v = \Delta$
    - When $v \notin I$, can still have $X_v = \Theta(\Delta)$
  - Easier to get useful bounds on X_v

Lemma 5.2.6

If $\Delta \geq 16$, we have $\mathbb{E}[X_v] \geq \frac{\log \Delta}{4}$ for every v.

Deducing the Theorem

Theorem 5.2.4 (Ajtai, Komlós, Szemerédi, 1980; Shearer, 1995)

If G is an n-vertex triangle-free graph with maximum degree Δ, then

$$\alpha(G) \geq \frac{n \log \Delta}{8\Delta}.$$

Proof

- If $\Delta \leq 15$, done by $\alpha(G) \geq \frac{n}{\Delta+1}$
- Otherwise, let I be a uniformly random independent set of G
- For each vertex v, let $X_v = \Delta \cdot 1_{\{v \in I\}} + |I \cap N(v)|$
- Let $X = \sum_v X_v$
- Observe: $X \leq 2\Delta|I|$
  - Each $v \in I$ contributes at most 2Δ: Δ via X_v, and 1 via X_u for each neighbour u
- Lemma 5.2.6 $\Rightarrow \mathbb{E}[X] \geq \frac{n \log \Delta}{4}$ ∎

Proving the Lemma

Lemma 5.2.5

If $\Delta \geq 16$, we have $\mathbb{E}[X_v] \geq \frac{\log \Delta}{4}$ for every v.

Proof

- $X_v = \Delta \cdot 1_{\{v \in I\}} + |I \cap N(v)|$
- Which $u \in N(v)$ could be in I?
  - Need to know $I \cap N\big(N(v)\big)$
  - Idea: condition on how I meets the rest of the graph
  - Let $H = G \setminus \big(\{v\} \cup N(v)\big)$
- $\mathbb{E}[X_v] = \mathbb{E}\big[\mathbb{E}[X_v | I \cap V(H) = J]\big]$
- Suffices to show $\mathbb{E}[X_v | I \cap V(H) = J] \geq \frac{\log \Delta}{4}$ for every independent J in H

Extending Independent Sets

Goal

- $\mathbb{E}[X_v | I \cap V(H) = J] \geq \frac{\log \Delta}{4}$

Available neighbours

- Let $A = N(v) \setminus N(J)$
  - Those neighbours of v that could be added to J
- Let $a = |A|$

Independent extensions

- Two types of extensions of J to I:
  - $I = J \cup \{v\}$
  - $I = J \cup S$, some $S \subseteq A$
- I is chosen uniformly at random from $2^a + 1$ optoins

Computing Conditional Expectations

Recall

- $X_v = \Delta \cdot 1_{\{v \in I\}} + |I \cap N(v)|$
- Want to show $\mathbb{E}[X_v | I \cap V(H) = J] \geq \frac{\log \Delta}{4}$

Conditional Expectation

- Case: $v \in I$
  - Probability: $\frac{1}{2^a+1}$
  - $X_v = \Delta$
- Case: $v \notin I$
  - Probability: $\frac{2^a}{2^a+1}$
  - $\mathbb{E}[X_v | v \notin I, I \cap V(H) = J] = \mathbb{E}[|S|] = \frac{a}{2}$
- $\Rightarrow \mathbb{E}[X_v | I \cap V(H) = J] = \frac{\Delta}{2^a+1} + \frac{\mathrm{a}2^{\mathrm{a}-1}}{2^{\mathrm{a}}+1}$

Concluding Calculations

Recall

- $\mathbb{E}[X_v | I \cap V(H) = J] = \frac{\Delta}{2^a+1} + \frac{a2^{a-1}}{2^a+1}$
- Want to show $\mathbb{E}[X_v | I \cap V(H) = J] \geq \frac{\log \Delta}{4}$

Contradiction

- If not, $\frac{\log \Delta}{4} > \frac{\Delta}{2^a+1} + \frac{a2^{a-1}}{2^a+1}$
  - $\Rightarrow (2^a + 1) \log \Delta > 4\Delta + 2a2^a$
  - $\Rightarrow (\log \Delta - 2a)2^a > 4\Delta - \log \Delta$
  - Also $\Rightarrow a \geq 1$
- Must have $2a < \log \Delta$
  - $\Rightarrow 2^a < \sqrt{\Delta}$
- $\Rightarrow (\log \Delta - 2)\sqrt{\Delta} > 4\Delta - \log \Delta$
- False for $\Delta \geq 16$ $\blacksquare$

Epilogue

What we know

- $\Omega\left(\frac{k^2}{\log^2 k}\right) = R(3,k) = O\left(\frac{k^2}{\log k}\right)$

Theorem 5.2.7 (Kim, 1995)

As $k \to \infty$, $R(3,k) = \Omega\left(\frac{k^2}{\log k}\right)$.

Remarks

- Kim's proof a "tour de force"
- Lower bound recently sharpened via analysis of triangle-free process
- Asymptotics of $R(s,k)$, $s \geq 4$ fixed and $k \to \infty$, unknown

Any questions?

§3 Hamiltonicity

Chapter 5: Concentration

The Probabilistic Method

Setting the Scene

Definition 5.3.1

A Hamiltonian cycle in a graph G is a cycle passing through every vertex of G. A graph is called Hamiltonian if it contains a Hamiltonian cycle.

Theorem 5.3.2 (Karp, 1972)

Deciding whether a graph is Hamiltonian is NP-Complete.

Questions

- Are there easy ways to recognise Hamiltonian graphs?
- What happens for the average graph?

A Sufficient Condition

Theorem 5.3.3 (Dirac, 1952)

Every n-vertex graph G with minimum degree $\delta(G) \geq \frac{n}{2}$ is Hamiltonian.

Optimal bound

- n even: two disjoint cliques
- n odd: two cliques sharing one vertex

Corollary 5.3.4

For every $\varepsilon > 0$ and $p \geq \left(\frac{1}{2} + \varepsilon\right) n$, $G(n,p)$ is Hamiltonian w.h.p.

Threshold Lower Bound

First moment

- There are $\frac{(n-1)!}{2} = \left(\frac{n}{(1+o(1))e}\right)^n$ possible Hamiltonian cycles
- Each appears in $G(n,p)$ with probability p^n
- $\Rightarrow$ expected number of cycles is $\left(\frac{np}{(1+o(1))e}\right)^n$
- $\Rightarrow$ if $p \leq \frac{e-\varepsilon}{n}$, then $G(n,p)$ has no Hamiltonian cycles w.h.p.

Connectivity

- $G(n,p)$ Hamiltonian $\Rightarrow$ $G(n,p)$ connected

Proposition 5.3.5

For every $\varepsilon > 0$ and $p \leq \frac{(1-\varepsilon)\log n}{n}$, $G(n,p)$ is w.h.p. not Hamiltonian.

Dirac's Theorem

Theorem 5.3.3 (Dirac, 1952)

Every n-vertex graph G with minimum degree $\delta(G) \geq \frac{n}{2}$ is Hamiltonian.

Proof

- G is connected
  - If not, smaller component would not support minimum degree
- Let $P = v_0 v_1 v_2 \dots v_t$ be a longest path
  - $N(\{v_0, v_t\}) \subseteq P$, as otherwise path could be extended
- Pigeonhole: $\exists i$ such that $\{v_i, v_t\}, \{v_0, v_{i+1}\} \in E(G)$
- We have a cycle $C = v_0 v_1 v_2 \dots v_i v_t v_{t-1} v_{t-2} \dots v_{i+1} v_0$
- If $t = n$, this is a Hamiltonian cycle
- If $t < n$, connectivity $\Rightarrow$ edge from C to $G \setminus C$
  - Gives a longer path, contradiction. $\blacksquare$

Dirac's Algorithm

More than existential

- Proof shows us how to find a Hamiltonian cycle
- Start with any path
- If there are edges out from the endpoints, extend path
- Otherwise by pigeonhole turn path into cycle
  - Use external edge to extend path
- Repeat until cycle is Hamiltonian

Random setting

- Extremal problem:
  - Need to assume worst-case graph
  - Used large degree, pigeonhole to rotate path into cycle
- Can we use properties of $G(n, p)$ to do this more efficiently?

Pósa Rotations

Goal

- Given path $P = v_0 v_1 \dots v_t$ in a graph G
- Want to find a longer path or a Hamiltonian cycle

Definition 5.3.6 (Booster)

Given a graph G, a booster is a potential edge e such that $G \cup \{e\}$ contains a longer path or a Hamiltonian cycle.

Rotations

- If G is connected, the pair $\{v_0, v_t\}$ is a booster
- Suppose $\{v_i, v_t\} \in E(G)$, $1 \leq i \leq t-2$
  - Rotation along $\{v_i, v_t\}$: $P' = v_0 v_1 \dots v_i v_t v_{t-1} \dots v_{i+1}$ also a path of length t
  - $\Rightarrow$ the pair $\{v_0, v_{i+1}\}$ is also a booster

Endpoint Neighbourhoods

Lemma 5.3.7

Let $P = v_0 v_1 \dots v_t$ be a longest path in a graph G, and let R be the set of endpoints reachable from v_0 by sequences of rotations. Then $N_G(R) \subseteq N_P(R)$.

Proof

- After rotating along $\{v_i, v_t\}$, only v_i, v_t get new neighbours on the path
- Let $v \in R$
  - Rotate to path P' with v as an endpoint
- Let $y \in N(v) \setminus R$
  - If $y \notin V(P)$, extend P' to $y \Rightarrow$ longer path than P
  - If $y \in V(P)$, rotate P' along the edge $\{v, y\}$
  - $\Rightarrow$ a neighbour x of y on P' is an endpoint of the new path, so $x \in R$
  - If x also a neighbour of y on P, then $y \in N_P(R)$
  - Otherwise must have rotated along an edge incident to $y \Rightarrow y \in N_P(R)$ ■

Expanders

Corollary 5.3.8

Let P be a longest path in G, and let R be the set of endpoints following sequences of rotations. Then $|N_G(R)| \leq 2|R| - 1$.

Proof

- Lemma 5.3.7 $\Rightarrow N_G(R) \subseteq N_P(R)$
- Each vertex in R contributes at most two neighbours to $N_P(R)$
- Final vertex v_t only contributes one
- $\Rightarrow |N_P(R)| \leq 2|R| - 1$ ∎

Definition 5.3.9 (Expander)

A graph G is a $(k, 2)$-expander if, for every $S \subseteq V(G)$ with $|S| \leq k$, we have $|N_G(S)| \geq 2|S|$.

Expanders Have Many Boosters

Corollary 5.3.10

If G is a connected $(k, 2)$-expander, then G has at least $\frac{1}{2}k^2$ boosters.

Proof

- If G is Hamiltonian, every edge is a booster.
- Otherwise let $P = v_0 v_1 \dots v_t$ be a longest path
- Fix v_0, and let R_0 be the endpoints after rotations
  - Corollary 5.3.8 ⇒ $|N_G(R_0)| \leq 2|R_0| - 1$
- G a $(k, 2)$-expander ⇒ $|R_0| \geq k + 1$
- Given any $y \in R_0$, rotate to a v_0-y path P'
- Fix y, and let R_y be the endpoints of paths from y after rotating P'
  - Again, $|R_y| \geq k + 1$
- For each $z \in R_y$, $\{y, z\}$ is a booster, counted at most twice ∎

Dirac’s Algorithm in Random Graphs

Assumptions

- $G(n,p)$ is connected – know to be true for $p \geq \frac{(1+\varepsilon)\log n}{n}$
- $G(n,p)$ is a $(k,2)$-expander for k large

Rotation-Extension process

- Start with a longest path P
- Corollary 5.3.10 ⇒ gives rise to $\Omega(k^2)$ boosters
- Each booster is an edge of $G(n,p)$ independently with probability p
    - ⇒ Probability none of the boosters appear is $(1-p)^{k^2}$
    - ⇒ if $p = \omega(k^{-2})$, then w.h.p. one of the boosters should be in $G(n,p)$
- Use it to extend path, repeat until Hamiltonian

Multiple Exposures

Recall

- Longest path gave rise to $\Omega(k^2)$ boosters
- Want to show w.h.p. a booster appears in $G(n,p)$

Problem

- To find the boosters, we needed to expose edges in $\binom{V(P)}{2}$
  - Might already have found boosters are not edges
  - They do not appear independently with probability p

Solution

- Split the random graph into independent subgraphs
  - Let p_0, q satisfy $1-p=(1-p_0)(1-q)$
  - Then $G(n,p) \sim G(n,p_0) \cup G(n,q\ )$
- Use $G(n,p_0)$ to obtain connectivity, expansion properties, find boosters
- Use $G(n,q)$ to show boosters appear in the random graph w.h.p.

Random Graphs are Expanders

Lemma 5.3.11

If $p \geq \frac{7 \log n}{n}$, then $G(n,p)$ is w.h.p. an $\left(\frac{n}{6}, 2\right)$-expander.

Proof

- If not, there is some set S of size $s \coloneqq |S| \leq \frac{n}{6}$ such that $|N(S)| < 2s$
  - $\Rightarrow \exists W \subset V(G) \setminus S$, $|W| = 2s$, such that we have no edges from S to $V(G) \setminus (S \cup W)$
  - Probability these edges are missing is $(1-p)^{s(n-3s)} \leq e^{-ps(n-3s)} \leq e^{-psn/2}$
- Count number of pairs (S, W)
  - $\binom{n}{s} \leq \left(\frac{ne}{s}\right)^s$ choices for S, $\binom{n-s}{2s} \leq \binom{n}{2s} \leq \left(\frac{ne}{2s}\right)^{2s}$ choices for W
- Union bound
  - $\mathbb{P}(G(n,p)\text{ bad}) \leq \sum_{s=1}^{\frac{n}{6}} \left(\frac{n^3 e^3}{4s^3} e^{-pn/2}\right)^s \leq \sum_{s=1}^{\frac{n}{6}} \left(\frac{e^3}{4\sqrt{n}}\right)^s = o(1)$ ∎

The Hamiltonicity Threshold

Theorem 5.3.12 (Pósa, 1976)

If $p \geq \frac{80 \log n}{n}$, then $G(n,p)$ is w.h.p. Hamiltonian.

Proof

- Let $p_0 = \frac{7 \log n}{n}$ and $q = \frac{73 \log n}{n^2}$
- Let $G_0 \sim G(n, p_0)$, and for $i \in [n]$, let $G_i \sim G(n,q)$ be independent
  - If $G = G_0 \cup (\cup_i G_i)$, then $G \sim G(n,p)$ for $p = 1 - (1-p_0)(1-q)^n \leq \frac{80 \log n}{n^2}$
- Lemma 5.3.11 $\Rightarrow$ G_0 is w.h.p. a connected $\left(\frac{n}{6}, 2\right)$-expander
- Corollary 5.3.10 $\Rightarrow$ any supergraph of G_0 has at least $\frac{n^2}{72}$ boosters
  - $\Rightarrow$ probability G_i does not contain one of the boosters $\leq (1-q)^{\frac{n^2}{72}} \leq e^{\frac{-qn^2}{72}} = o\left(\frac{1}{n}\right)$
- $\Rightarrow$ Grow a longest path, using G_i to find a booster in the ith step $\blacksquare$

Epilogue

- Hamiltonicity displays a very sharp threshold

Theorem 5.3.13 (Komlós-Szemerédi, 1983)

For $\varepsilon > 0$ and $p \geq \frac{(1+\varepsilon)\log n}{n}$, $G(n,p)$ is w.h.p. Hamiltonian.

- Even sharper results were later proven

Theorem 5.3.14 (Bollobás, 1984; Ajtai-Komlós-Szemerédi, 1985)

In the random graph process, w.h.p. the graph becomes Hamiltonian precisely when the minimum degree is at least two.

Any questions?

§4 Martingales

Chapter 5: Concentration

The Probabilistic Method

Threshold for Triangles

Theorem 3.3.1

For $\ell \geq 2$, the threshold for $K_\ell \subseteq G(n, p)$ is $p_0(n) = n^{-2/(\ell-1)}$.

Triangular case

- $\ell = 3$: threshold for containing triangles is n^{-1}

Upper tail

- When $p \gg n^{-1}$, how unlikely is $G(n, p)$ to be triangle-free?
- Proof of Theorem 3.3.1
  - Used Chebyshev's Inequality
  - Gives polynomial error bounds

Exponential Dreams

Indicator random variables

- Let X denote the number of triangles in $G \sim G(n,p)$
- Given $T \in \binom{[n]}{3}$, let X_T be the indicator that $G[T] \cong K_3$
  - Then $\mathbb{P}(X_T = 1) = p^3$
  - Also $X = \sum_T X_T$

Stronger concentration

- Using Chernoff would give $\mathbb{P}(X = 0) \leq \exp\left(-\frac{1}{2}\binom{n}{3}p^3\right)$
  - Exponentially small error bound
- Problem: summands X_T not independent
  - $X_T, X_{T'}$ positively correlated when $|T \cap T'| = 2$

Sparse Independence

Cheap fix

- Restrict our attention to mutually independent events
- Equivalently: consider a family of edge-disjoint triangles

Lemma 5.4.1

There exists a family of $\frac{1}{3}\binom{n-1}{2}$ pairwise edge-disjoint triangles in K_n.

Proof

- Colour each triangle $\{i, j, k\}$ with the colour $c \equiv i + j + k \pmod{n}$
- Each colour class is edge-disjoint
  - Given vertices i, j, third vertex $k \equiv c - i - j$ determined
- Large colour class
  - For some c, number of c-coloured triangles is at least $\frac{1}{n}\binom{n}{3} = \frac{1}{3}\binom{n-1}{2}$ ∎

Don’t Let Your Dreams Be Dreams

Corollary 5.4.2

$G(n,p)$ is triangle-free with probability at most $\exp\left(-\frac{1}{3}\binom{n-1}{2}p^3\right)$.

Proof

- Let $\mathcal{T}$ be the collection of triangles from Lemma 5.4.1
- If $G(n,p)$ is triangle-free, no triangle in $\mathcal{T}$ appears
  - These appear independently
- Probability none appear is $(1-p^3)^{|\mathcal{T}|} \le \exp(-|\mathcal{T}|p^3)$ $\blacksquare$

Good news

- Exponential bound on error probability

Bad news

- Exponent $\binom{n-1}{2}p^3 = \Theta(n^2p^3)$ is of lower order than expected

Postmortem of a Proof

Improving the exponent

- Need to consider all $\binom{n}{3}$ possible triangles
- Dependencies are limited – can we recover Chernoff-type bounds?

Revisiting Chernoff

- $S_n = \sum_{i=1}^{n} X_i$
- Properties of X_i:
  - Bounded, $\{-1,1\}$-variables
  - $\mathbb{E}[X_i] = 0$
  - X_i mutually independent
- Using independence:
  - Applied Markov to e^{S_n}
  - Independence $\Rightarrow \mathbb{E}[e^{S_n}] = \mathbb{E}\left[e^{\sum_i X_i}\right] = \prod_i \mathbb{E}[e^{X_i}]$

Martingales

Conditional independence

- What if the X_i are not independent?
  - Recover independence by conditioning on previous variables
- Product rule: $\mathbb{E}\left[e^{\sum_i X_i}\right] = \prod_i \mathbb{E}\left[e^{X_i} \middle| \{X_j : j < i\}\right]$
- $\Rightarrow$ if $\left(X_i \middle| \{X_j : j < i\}\right)$ has the right properties, can prove Chernoff-type bounds

Definition 5.4.3 (Martingale)

A *martingale* is a sequence $Z_0, Z_1, \dots, Z_m$ of random variables such that, for each $1 \leq i \leq m$, we have

$$\mathbb{E}\left[Z_i \middle| \{Z_j : j < i\}\right] = Z_{i-1}.$$

Loosely speaking, given what has previously transpired, we expect nothing to change in the ith step.

Martingales, Tame and Wild

Boring mathsy example

- Let X_i be independent and uniform on $\{-1,1\}$, for $1 \le i \le m$
- Let $Z_i = \sum_{j \le i} X_j$
- $\mathbb{E}\left[Z_i \middle| \{Z_j : j < i\}\right] = \mathbb{E}\left[Z_{i-1} + X_i \middle| \{Z_j : j < i\}\right] = Z_{i-1} + \mathbb{E}\left[X_i \middle| \{Z_j : j < i\}\right]$
  - $\mathbb{E}\left[X_i \middle| \{Z_j : j < i\}\right] = \mathbb{E}[X_i] = 0$
  - $\Rightarrow (\mathrm{Z_i} : 0 \le i \le m)$ is a martingale

Fun real-world example

- Gambling on (fair) coin tosses
- $Z_i =$ cumulative profit/loss after ith toss
- Bet $b_i = b_i(Z_0, Z_1, \dots, Z_{i-1})$ on the ith toss, depending on previous outcomes
- $\mathbb{E}\left[Z_i \middle| \{Z_j : j < i\}\right] = \frac{1}{2}(\mathrm{Z_{i-1}} + \mathrm{b_i}) + \frac{1}{2}(\mathrm{Z_{i-1}} - \mathrm{b_i}) = \mathrm{Z_{i-1}}$
  - $\Rightarrow (Z_i : 0 \le i \le m)$ is a martingale

Disclaimer: gambling can be addictive and bad for your bank balance

Martingale Concentration

Theorem 5.4.4 (Azuma's Inequality)

Let $Z_0, Z_1, \dots, Z_m$ be a martingale with $Z_0 = 0$ and $|Z_i - Z_{i-1}| \leq 1$ for all $1 \leq i \leq m$. Then, for any a > 0, we have

$$\mathbb{P}(Z_m \geq a) \leq \exp(-a^2/2m).$$

Proof

- Set $X_i = Z_i - Z_{i-1}$
  - $\Rightarrow |X_i| \leq 1$ and $Z_m = \sum_{i=1}^m X_i$
  - Martingale $\Rightarrow \mathbb{E}\left[X_i \middle| \{Z_j : j < i\}\right] = 0$
- For any $\lambda > 0$, we have $\{Z_m \geq a\} \Leftrightarrow \left\{\mathrm{e}^{\lambda \mathrm{Z_m}} \geq e^{\lambda a}\right\}$
- $\mathbb{P}\left(e^{\lambda Z_m} \geq e^{\lambda a}\right) \leq \mathbb{E}\left[e^{\lambda Z_m}\right] e^{-\lambda a}$
- $\mathbb{E}\left[e^{\lambda Z_m}\right] = \prod_{i=1}^m \mathbb{E}\left[e^{\lambda X_i} \middle| \{Z_j : j < i\}\right]$

A Little Calculus

Lemma 5.4.5

If $\lambda > 0$ and Y is a random variable with $\mathbb{E}[Y] = 0$ and $|Y| \leq 1$, then

$$\mathbb{E}\left[e^{\lambda Y}\right] \leq \cosh \lambda \,.$$

Proof

- Let $f(y) = \frac{e^{\lambda}+e^{-\lambda}}{2} + \frac{e^{\lambda}-e^{-\lambda}}{2} y = e^{\lambda}\left(\frac{1}{2} + \frac{y}{2}\right) + e^{-\lambda}\left(\frac{1}{2} - \frac{y}{2}\right)$
  - $\Rightarrow f$ represents chord between $g(y) = e^{\lambda y}$ between $y = -1$ and $y = 1$
- Convexity $\Rightarrow g(y) \leq f(y)$ for all $y \in [-1,1]$
- Thus $\mathbb{E}\left[e^{\lambda Y}\right] = \mathbb{E}[g(Y)] \leq \mathbb{E}[f(Y)] = \frac{e^{\lambda}+e^{-\lambda}}{2} + \frac{e^{\lambda}-e^{-\lambda}}{2} \mathbb{E}[Y] = \cosh \lambda$ ■

Completing the Proof

Theorem 5.4.4 (Azuma's Inequality)

Let $Z_0, Z_1, \ldots, Z_m$ be a martingale with $Z_0 = 0$ and $|Z_i - Z_{i-1}| \leq 1$ for all $1 \leq i \leq m$. Then, for any a > 0, we have

$$\mathbb{P}(Z_m \geq a) \leq \exp(-a^2/2m).$$

Proof (cont'd)

- $\mathbb{P}\left(e^{\lambda Z_m} \geq e^{\lambda a}\right) \leq \mathbb{E}\left[e^{\lambda Z_m}\right] e^{-\lambda a}$
- $\mathbb{E}\left[e^{\lambda Z_m}\right] = \prod_{i=1}^{m} \mathbb{E}\left[e^{\lambda X_i} \middle| \{Z_j : j < i\}\right]$
- By Lemma 5.4.5, $\mathbb{E}\left[e^{\lambda X_i} \middle| \{Z_j : j < i\}\right] \leq \cosh \lambda \leq e^{\lambda^2/2}$
- $\therefore \mathbb{P}(Z_m \geq a) \leq \exp\left(\frac{\lambda^2 m}{2} - \lambda a\right)$
- Substitute $\lambda = \frac{a}{m}$

∎

Graph Martingales

Upper tail for triangles

- Sample $G \sim G(n,p)$, $X =$ # triangles in G
- $X = \sum_{T \in \binom{[n]}{3}} X_T$, with X_T the indicator that $G[T] \equiv K_3$

Where is the martingale?

- Natural candidate
  - Order sets $T_1, T_2, \dots, T_m$
  - Let $Z_i = \sum_{j \leq i} X_{T_j}$
- Problem
  - Positive correlation $\Rightarrow$ cannot make $\mathbb{E}\left[X_{T_i} \middle| \{Z_j : j < i\}\right] = 0$ for all choices of Z_j
- Solution
  - Reveal information about G in stages
  - Let Z_i be the expected value of X given the information after i rounds

The Doob Martingale

General framework

- Sample $G \sim G(n,p)$, interested in graph parameter $f(G) \in \mathbb{R}$
  - Example: $f(G) = \#$ triangles in G

Revealing G

- Order the possible edges $\binom{[n]}{2} = \{e_1, e_2, \dots, e_m\}$ for $m = \binom{n}{2}$
  - Let $S_i = \{e_j : j \le i\}$

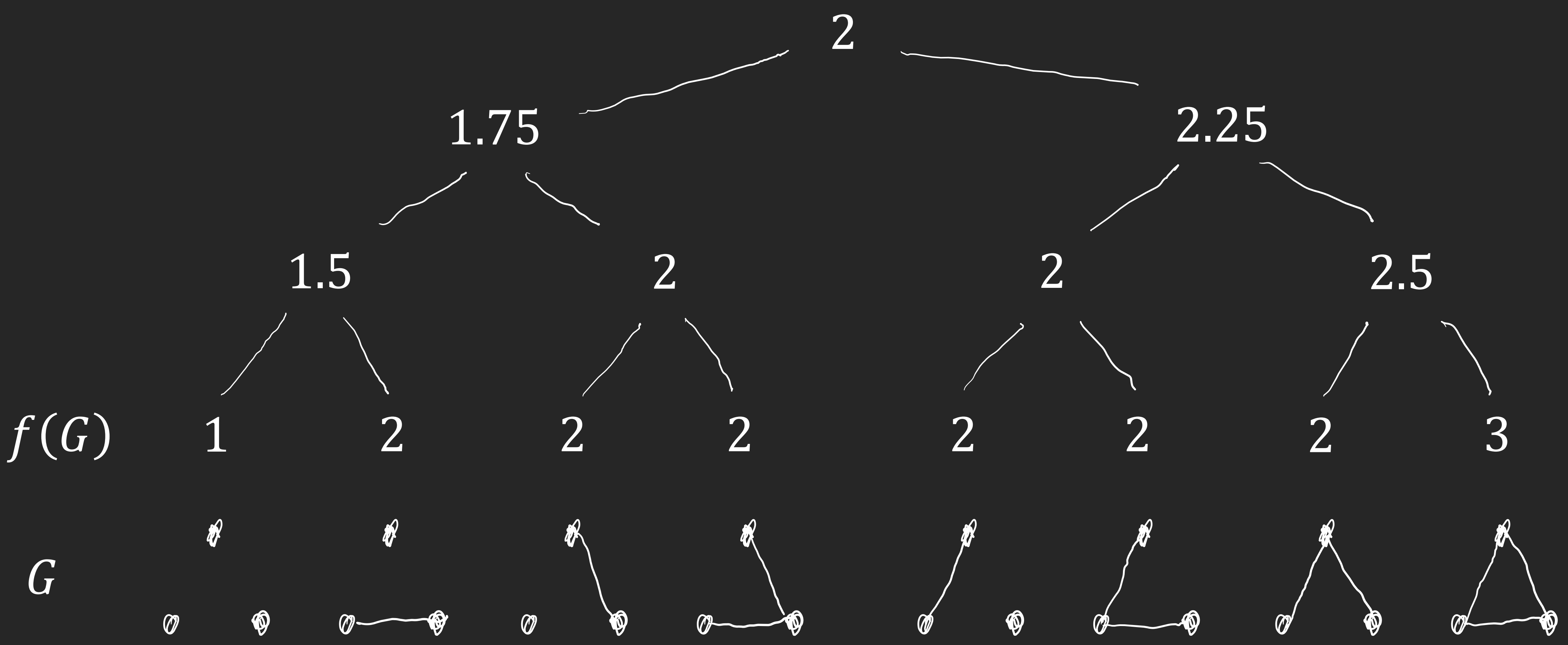
The martingale

- $Z_i = \mathbb{E}[f(G) | E(G) \cap S_i] - \mathbb{E}[f(G)]$
  - Expected value of parameter given the previously revealed edges
- $Z_0 = \mathbb{E}[f(G)] - \mathbb{E}[f(G)] = 0$
- $Z_m = \mathbb{E}\left[f(G) \middle| E(G) \cap \binom{[n]}{2}\right] - \mathbb{E}[f(G)] = f(G) - \mathbb{E}[f(G)]$

A Small Example

Framework

- $G \sim G\left(3, \frac{1}{2}\right)$ and $f(G) = \omega(G)$



Verifying Martingale-ness

Recall

- $G \sim G(n,p)$, and we are exploring a graph parameter $f(G)$
- $S_i = \{e_j : j \le i\}$
- $Z_i = \mathbb{E}[f(G) | E(G) \cap S_i]$

Conditional expectations

- $\mathbb{E}[Z_{i+1} | E(G) \cap S_i] = \mathbb{E}[\mathbb{E}[f(G) | E(G) \cap S_{i+1}] | E(G) \cap S_i]$
  $= \mathbb{E}[f(G) | E(G) \cap S_i] = Z_i$
- $\Rightarrow$ this is a martingale

Lipschitz Properties

Bounded differences

- To apply Azuma's Inequality, we need $|Z_i - Z_{i-1}| \leq 1$ for all i
- Intuitively: changing one edge should not change $f(G)$ much

Definition 5.4.6 (c-Lipschitz)

Let $c > 0$. A graph parameter f is *c-(edge-)Lipschitz* if, for any edge e, $|f(G) - f(G \triangle e)| \leq c$.

Fact 5.4.7

Given a c-Lipschitz parameter f, we have $|Z_i - Z_{i-1}| \leq 1$ for the normalised Doob martingale $Z_i = \frac{1}{c}(\mathbb{E}[f(G)|E(G) \cap S_i] - \mathbb{E}[f(G)])$.

Summary

Theorem 5.4.4 (Azuma's Inequality)

Let $Z_0, Z_1, \dots, Z_m$ be a martingale with $Z_0 = 0$ and $|Z_i - Z_{i-1}| \le 1$ for all $1 \le i \le m$. Then, for any a > 0, we have

$$\mathbb{P}(Z_m \ge a) \le \exp(-a^2/2m)\,.$$

Corollary 5.4.8

Let f be a c-Lipschitz graph parameter, $G \sim G(n,p)$, $\mu = \mathbb{E}[f(G)]$, and $a > 0$. Then $\mathbb{P}(f(G) \ge \mu + a) \le \exp(-a^2/n^2c^2)\,.$

Remarks

- Same bound holds for $\mathbb{P}(f(G) \le \mu - a)$
- Can also use a vertex-exposure martingale
  - Z_i is the expected value of $f(G)$ after exposing induced subgraph of G on first i vertices

Any questions?

§5 Triangle-free Graphs

Chapter 5: Concentration

The Probabilistic Method

A Quick Review

Theorem 3.3.1

For $\ell \geq 2$, the threshold for $K_\ell \subseteq G(n,p)$ is $p_0(n) = n^{-2/(\ell-1)}$.

Triangle-freeness

- $\Rightarrow$ when $p = \omega(n^{-1})$, $\mathbb{P}\big(K_3 \nsubseteq G(n,p)\big) = o(1)$
- Error bound from Chebyshev $\Rightarrow$ only polynomially small

Exponential error bounds

- Sharper estimates by considering edge-disjoint triangles

Corollary 5.4.2

$G(n,p)$ is triangle-free with probability at most $\exp\left(-\frac{1}{3}\binom{n-1}{2}p^3\right)$.

Applying Azuma

Corollary 5.4.8’

Let f be a c-Lipschitz graph parameter, $G \sim G(n,p)$, $\mu = \mathbb{E}[f(G)]$, and $a > 0$. Then $\mathbb{P}(f(G) \le \mu - a) \le \exp(-a^2/n^2c^2)$.

Counting triangles

- $f(G) =$ # triangles in G
- $\mu = \binom{n}{3}p^3 = \mathrm{a}$
- $c = n - 2$

Corollary 5.5.1

$G(n,p)$ is triangle-free with probability at most $\exp\left(\frac{-(n-1)^2p^6}{36}\right)$.

Immeasurable Disappointment

Worse exponent

- Exponent $\frac{1}{36}(n-1)^2p^6$ is worse than the $\frac{1}{3}\binom{n-1}{2}p^3$ from before
- Problems
  - Long martingale, $\binom{n}{2}$, and large Lipschitz constant, $n-2$
- What if we apply vertex-exposure instead?

Corollary 5.5.2

Let f be a c_v-vertex-Lipschitz parameter, $\mu = \mathbb{E}[f(G)]$, and $a > 0$. Then, for $G \sim G(n,p)$, $\mathbb{P}(f(G) \leq \mu - a) \leq \exp(-a^2/2nc_v^2)$.

Vertex-exposure martingale

- Shorter martingale, n, but worse Lipschitz constant, $\binom{n-1}{2}$
- Yields a worse exponent, $\Theta(np^6)$

A Judicious Parameter

Reducing the Lipschitz constant

- Need to decrease the influence a single edge can have
  - Idea: edge-disjoint triangles
- Let $f(G) =$ maximum number of pairwise edge-disjoint triangles

Corollary 5.4.8

Let f be a c-Lipschitz graph parameter, $G \sim G(n,p)$, $\mu = \mathbb{E}[f(G)]$, and $a > 0$. Then $\mathbb{P}(f(G) \geq \mu + a) \leq \exp(-a^2/n^2c^2)$.

New bound

- This choice of f is 1-Lipschitz
- Still have G triangle-free $\Leftrightarrow f(G) = 0$, so take $a = \mathbb{E}[f(G)]$
- $\Rightarrow \mathbb{P}(G \text{ triangle}-\text{free}) \leq \exp(-\mathbb{E}[f(G)]^2/n^2)$
  - How do we bound this expectation?

Edge-Disjoint Triangles

Lemma 5.5.3

Let $q \in [0,1]$, and let G be a graph with X triangles and Y pairs of triangles sharing an edge. Then G has a collection of m pairwise edge-disjoint triangles, for some $m \geq qX - q^2Y$.

Proof

- Let $\mathcal{T}$ be the collection of all X triangles in G
- Let $\mathcal{R}' \subseteq \mathcal{T}$ be a q-random subcollection
  - Triangle $T \in \mathcal{R}'$ with probability q, independent of all other triangles
- Let Y' be the number of pairs of overlapping triangles in $\mathcal{R}'$
- From each pair in $\mathcal{R}'$ sharing an edge, remove one of the triangles
  - $\Rightarrow$ resulting $\mathcal{R} \subseteq \mathcal{R}'$ is pairwise edge-disjoint
- $\mathbb{E}[|\mathcal{R}|] \geq \mathbb{E}[|\mathcal{R}'| - Y'] = qX - q^2Y$ ∎

Random Edge-Disjoint Triangles

Random graph setting

- Let $G \sim G(n,p)$, $X =$ # triangles, $Y =$ # overlapping pairs of triangles
- Lemma 5.5.3 $\Rightarrow f(G) \geq qX - q^2Y$ for all $q \in [0,1]$
  - $\Rightarrow \mathbb{E}[f(G)] \geq q\mathbb{E}[X] - q^2\mathbb{E}[Y]$

Choosing values

- We have $\mathbb{E}[X] = \binom{n}{3}p^3$, $\mathbb{E}[Y] = \binom{n}{2}\binom{n-2}{2}p^5$
- Calculus $\Rightarrow$ optimal $q = \frac{1}{3np^2}$

Corollary 5.5.4

Let $G \sim G(n,p)$ for $p \geq \frac{1}{\sqrt{3n}}$. Then $\mathbb{E}[f(G)] \geq \left(\frac{1}{36} - o(1)\right) n^2 p$.

Immeasurable Joy

Recall

- $G \sim G(n,p)$
- $f(G) =$ maximum number of pairwise edge-disjoint triangles in G
- Corollary 5.5.4 ⇒ if $p \geq 1/\sqrt{3n}$, then $\mathbb{E}[f(G)] \geq \Omega(n^2 p)$
- Corollary 5.4.8 ⇒ $\mathbb{P}(G \text{ triangle}-\text{free}) \leq \exp(-\mathbb{E}[f(G)]^2/n^2)$

Theorem 5.5.5

Let $p \geq \frac{1}{\sqrt{3n}}$ and let $G \sim G(n,p)$. Then

$$\mathbb{P}(K_3 \not\subseteq G) \leq \exp\left(-\Omega(n^2 p^2)\right).$$

- Improves previous exponent when $cn^{-1/2} \leq p \leq \mathrm{c'n}$

Any questions?

§6 Chromatic Number

Chapter 5: Concentration

The Probabilistic Method

Introducing the Problem

General bounds

- What makes the chromatic number large?
- $\chi(G) \geq \omega(G)$
- $\chi(G) \geq \frac{n}{\alpha(G)}$

Complexity

- Determining chromatic number of graphs is NP-Complete
- Even deciding if a graph is 3-colourable is NP-Complete

Typical behaviour

- What can we say about $\chi\left(G\left(n, \frac{1}{2}\right)\right)$?

Colouring Random Graphs

Question

- What is $\chi\left(G\left(n,\frac{1}{2}\right)\right)$?

Applying general bounds

- Homework: with high probability, $\omega\left(G\left(n,\frac{1}{2}\right)\right) \sim 2\log n$
  - $\Rightarrow \chi\left(G\left(n,\frac{1}{2}\right)\right) \geq (2+o(1))\log n$
- Symmetry $\Rightarrow \alpha\left(G\left(n,\frac{1}{2}\right)\right) \sim 2\log n$
  - $\Rightarrow \chi\left(G\left(n,\frac{1}{2}\right)\right) \geq \frac{(1+o(1))n}{2\log n}$
  - Homework: will show this bound is sharp

Honing In

- Can we further narrow down the likely values of $\chi\left(G\left(n,\frac{1}{2}\right)\right)$?

Lemma 5.6.1

The parameter $\chi(G)$ is 1-vertex-Lipschitz.

Proof

- Let $v \in V(G)$ be arbitrary, and let $H = G[V \setminus \{v\}]$
- Chromatic number is monotone increasing
  - $\Rightarrow \chi(G) \geq \chi(H)$
- Can always assign v a new colour
  - $\Rightarrow \chi(G) \leq \chi(H) + 1$
- $\Rightarrow$ changing G at v can change $\chi(G)$ by at most one $\blacksquare$

Colouring with Martingales

Theorem 5.6.2

For $\varepsilon > 0$ there is a constant $C = C(\varepsilon)$ such that for every n there is an interval $I_n \subseteq [n]$ of length $C\sqrt{n}$ such that, for $G \sim G\left(n, \frac{1}{2}\right)$,

$$\mathbb{P}(\chi(G) \notin I_n) \leq \varepsilon.$$

Proof

- Apply the vertex-exposure martingale to the parameter $\chi(G)$
  - $Z_i = \mathbb{E}\big[\chi(G)\big|G[[i]]\big] - \mathbb{E}[\chi(G)], 0 \leq i \leq n$
- Lemma 5.6.1: $\chi(G)$ is 1-vertex-Lipschitz
- Azuma's inequality: $\mathbb{P}(|Z_i| \geq a) \leq 2\exp(-a^2/2n)$
- If $a = \sqrt{2n\ln\frac{2}{\varepsilon}}$, right-hand size is ε
- $\Rightarrow$ can take $I_n = (\mu - a, \mu + a)$, where $\mu = \mathbb{E}[G]$ ■

Reflections on our Results

Narrow window

- Previously saw that $\chi(G) \geq \frac{(1+o(1))n}{2 \log n} = n^{1-o(1)}$ almost surely
  - $\Rightarrow$ margin of error of $O(\sqrt{n})$ is relatively small
- Theorem doesn't say anything about *where* this interval is

Sparse random graphs

- Never used that $G \sim G\left(n, \frac{1}{2}\right)$
  - Proof applies to $G \sim G(n, p)$ for any $p = p(n)$
- However, result is trivial for sparse graphs
  - e.g.: if $p = o\left(\frac{1}{n}\right)$, then G is bipartite with high probability
  - If $p \leq \frac{c}{\sqrt{n}}$, then with high probability $\Delta(G) \leq C\sqrt{n} \Rightarrow \chi(G) \leq C\sqrt{n} + 1$

Colouring Subgraphs of Sparse Graphs

Proposition 5.6.3

Fix $\alpha > \frac{5}{6}$ and $c > 0$. Then, if $p = n^{-\alpha}$ and $G \sim G(n, p)$, with high probability G has the property that, for every set S of $c\sqrt{n}$ vertices, $\chi(G[S]) \leq 3$.

Proof

- If H is d-degenerate, then $\chi(H) \leq d + 1$
  - $\Rightarrow$ if $\chi(G[S]) > 3$ for some S, $G[S]$ is *not* 2-degenerate
- $\Rightarrow$ G contains some subgraph H with $v(H) \leq c\sqrt{n}$ and $\delta(H) \geq 3$
  - $\Rightarrow e(H) \geq \frac{3}{2} v(H)$
- Hence it suffices to show G is unlikely to contain such a subgraph

Subgraphs of Sparse Random Graphs are Sparse

Goal

- Every subgraph $H \subseteq G$ on at most $c\sqrt{n}$ vertices has at most $\frac{3v(H)}{2}$ edges

Proof (cont'd)

- Number of choices for $V(H)$: $\binom{n}{t} \leq \left(\frac{ne}{t}\right)^t$
- Number of choices of $\frac{3v(H)}{2}$ edges of H: $\binom{\binom{t}{2}}{\frac{3t}{2}} \leq \left(\frac{\binom{t}{2}e}{\frac{3t}{2}}\right)^{\frac{3t}{2}} \leq \left(\frac{te}{3}\right)^{\frac{3t}{2}}$
- $\Rightarrow \mathbb{P}(\exists \text{ bad } H \text{ on } t \text{ vertices}) \leq \left(\frac{ne}{t}\right)^t \left(\frac{te}{3}\right)^{\frac{3t}{2}} p^{\frac{3t}{2}} \leq \left(en^{1-3\alpha/2}t^{1/2}\right)^t$
- Since $t < c\sqrt{n}$, this is at most $\left(c'n^{5/4-3\alpha/2}\right)^t$
- As $\alpha > \frac{5}{6}$, exponent of n is negative
- $\Rightarrow$ summing over all t, $\mathbb{P}(\exists \text{ bad } H) = o(1)$ $\blacksquare$

Wow, Much Precise

Theorem 5.6.4 (Shamir-Spencer, 1987)

Fix $\alpha > \frac{5}{6}$ and set $p = n^{-\alpha}$. There is some $u = u(n,p)$ such that if $G \sim G(n,p)$, then almost surely $u \leq \chi(G) \leq u + 3$.

Proof idea

- Enough to focus on likely values of $\chi(G)$
- Consider the smallest u such that $\mathbb{P}(\chi(G) \leq u) = \Omega(1)$
- Show that one can colour *most* vertices of G with u colours
- Use Proposition 5.6.3 for the rest

A Wise Choice of Graph Parameter

Theorem 5.6.4

Fix $\alpha > \frac{5}{6}$ and set $p = n^{-\alpha}$. There is some $u = u(n,p)$ such that if $G \sim G(n,p)$, then almost surely $u \leq \chi(G) \leq u + 3$.

Proof

- Suffices to show that for any $\varepsilon > 0$, there is $u = u(n,p,\varepsilon)$ such that $\mathbb{P}(u \leq \chi(G) \leq u+3) \geq 1 - 3\varepsilon$
- Define $u = u(n,p,\varepsilon)$ to be smallest u such that $\mathbb{P}(\chi(G) \leq u) \geq \varepsilon$
  - $\Rightarrow \mathbb{P}(\chi(G) \leq u-1) < \varepsilon$
- Now wish to show that most vertices can be u-coloured
- Define $f(G) =$ minimum size of $S \subseteq V(G)$ such that $\chi(G[V \setminus S]) \leq u$
- $\mathbb{P}(f(G) = 0) = \mathbb{P}(\chi(G) \leq u) \geq \varepsilon$

Setting Up Azuma

Recall

- u: least integer such that $\mathbb{P}(\chi(G) \leq u) \geq \varepsilon$
- $f(G)$: minimum size of S such that $\chi(G[V \setminus S]) \leq u$

Lipschitz

- Fix a vertex $v \in V(G)$
- Choose a minimum set S' whose removal from $G[V \setminus \{v\}] \Rightarrow \chi \leq u$
- Worst-case: can always take $S = S' \cup \{v\}$
- $\Rightarrow f$ is 1-vertex-Lipschitz

Martingale

- Run the vertex-exposure martingale on $f(G)$

Completing the Proof

Recall

- $f(G)$: minimum size of S such that $\chi(G[V \setminus S]) \leq u$; let $\mu = \mathbb{E}[f(G)]$
- $\mathbb{P}(f(G) = 0) \geq \varepsilon$

Concentration

- Azuma's Inequality $\Rightarrow \mathbb{P}(f(G) \leq \mu - a) \leq \exp(-a^2/2n)$
  - $\Rightarrow \varepsilon \leq \mathbb{P}(f(G) = 0) \leq \exp(-\mu^2/2n)$
  - $\Rightarrow \mu \leq \sqrt{2n \ln 1/\varepsilon}$
- Azuma's Inequality $\Rightarrow \mathbb{P}(f(G) \geq \mu + a) \leq \exp(-a^2/2n)$
  - $\Rightarrow \mathbb{P}\left(f(G) \geq \mu + \sqrt{2n \ln 1/\varepsilon}\right) \leq \varepsilon$
  - $\Rightarrow \mathbb{P}\left(f(G) \geq 2\sqrt{2n \ln 1/\varepsilon}\right) \leq \varepsilon$

And voila

- $\Rightarrow$ can remove $c\sqrt{n}$ vertices and u colour the rest
- Proposition 5.6.3 $\Rightarrow$ can 3-colour removed vertices with probability $1 - \varepsilon$ $\blacksquare$

Epilogue

Location of interval

- Again, proof only shows concentration
  - Actual value of chromatic number not needed
- Concern: didn't our choice of u depend on ε?
  - Suppose $u = u(n, p, \varepsilon)$ and $u' = u(n, p, \varepsilon')$
  - We proved $\mathbb{P}(\chi(G) \in [u, u+3]) \geq 1 - \varepsilon$, $\mathbb{P}(\chi(G) \in [u', u'+3]) \geq 1 - \varepsilon'$
  - $\Rightarrow \mathbb{P}(\chi(G) \in [u, u+3] \cap [u', u'+3]) \geq 1 - \varepsilon - \varepsilon'$
  - $\Rightarrow$ Different u's give an even stronger concentration inequality

Further results

- Alon-Krivelevich (1997): if $\alpha > \frac{1}{2}$ and $p = n^{-\alpha}$, there is some $u = u(n, p)$ such that $\chi\big(G(n, p)\big) \in \{u, u+1\}$ with high probability
- Heckel-Riordan (2020+): if $I \subseteq [n]$ is an interval such that $\chi\big(G(n, 1/2)\big) \in I$ with high probability, then $|I| = n^{1/2 - o(1)}$

Any questions?