Chapter 7: Expectation Thresholds

The Probabilistic Method

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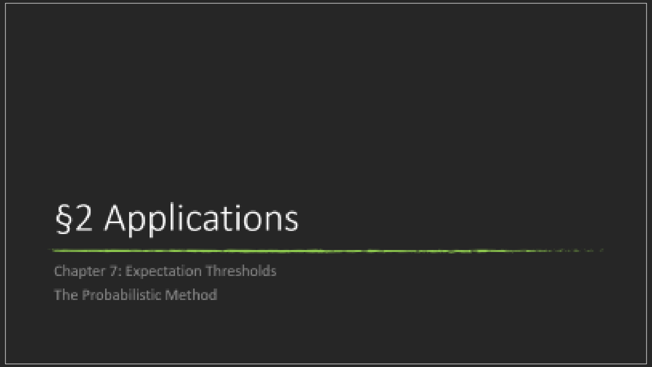
Chapter Overview

- Introduce (fractional) expectation thresholds
- Demonstrate applications of Talagrand's Conjecture



§1 Talagrand's Conjecture

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Thresholds Revisited

Definition 3.2.2 (Thresholds)

Given a nontrivial monotone graph property $\mathcal{P}$, $p_0(n)$ is a threshold for $\mathcal{P}$ if

$$\mathbb{P}(G(n,p) \in \mathcal{P}) \to \begin{cases} 0 \text{ if p} \ll p_0(n), \\ 1 \text{ if } p \gg p_0(n). \end{cases}$$

Theorem 3.2.4 (Bollobás-Thomason, 1987)

Every nontrivial monotone graph property has a threshold.

- Proof $\Rightarrow$ can take p_0 so that $\mathbb{P}(G(n,p) \in \mathcal{P}) = \frac{1}{2}$
- Issue: hard to calculate $\mathbb{P}(G(n,p) \in \mathcal{P})$ precisely

First Moment Estimates

Computing the probabilities

- $\mathbb{P}(G(n,p) \in \mathcal{P}) = \sum_{F \in \mathcal{P}} \mathbb{P}(G(n,p) = F) = \sum_{F \in \mathcal{P}} p^{e(F)}(1-p)^{\binom{n}{2}-e(F)}$
- Requires knowledge of sizes of graphs F with property $\mathcal{P}$
  - Even then, hard to determine when $\mathbb{P}(G(n,p) \in \mathcal{P}) = \frac{1}{2}$

Estimation

- $G \in \mathcal{P}$ if and only if G contains a *minimal* graph in $\mathcal{P}$
- Let $\mathcal{P}_0 = \{F \in \mathcal{P}: \forall H \subsetneq F, H \notin \mathcal{P}\}$
  - $G \in \mathcal{P} \Rightarrow \exists F \in \mathcal{P}_0 : F \subseteq G$
- Union bound
  - $\mathbb{P}(G(n,p) \in \mathcal{P}) = \mathbb{P}\big(\exists F \in \mathcal{P}_0 : F \subseteq G(n,p)\big) \leq \sum_{F \in \mathcal{P}_0} \mathbb{P}\big(F \subseteq G(n,p)\big)$
  - $\mathbb{P}\big(F \subseteq G(n,p)\big) = p^{e(F)}$
  - $\Rightarrow \mathbb{P}(G(n,p) \in \mathcal{P}) \leq \sum_{F \in \mathcal{P}_0} p^{e(F)}$
- Right-hand side is the expected number of minimal $\mathcal{P}$-graphs in $G(n,p)$

Threshold Lower Bound

First moment bound

- $\mathbb{P}(G(n,p) \in \mathcal{P}) \leq \sum_{F \in \mathcal{P}_0} p^{e(F)} \Rightarrow p_0 \geq p_1(\mathcal{P}) \coloneqq \max\left\{p: \sum_{F \in \mathcal{P}_0} p^{e(F)} \leq \frac{1}{2}\right\}$
- Bound is easier to compute
  - Only involves minimal graphs from $\mathcal{P}$, doesn't have pesky $(1-p)^{\binom{n}{2}-e(F)}$ terms

Tightness?

- Subgraph containment
  - $\mathcal{P} = \{G: H \subseteq G\} \Rightarrow \mathcal{P}_0 = \{\text{copies of } H \text{ on vertices } [n]\}$
  - $\Rightarrow p_1(\mathcal{P}) = \Omega\left(\mathrm{n}^{-\mathrm{v(H)/e(H)}}\right)$
  - Bound is tight for balanced graphs, e.g. cliques, but not in general
- Hamiltonicity
  - $\mathcal{P}_0 = \{\text{Hamiltonian cycles on } [n]\} \Rightarrow p_1(\mathcal{P}) = \Omega(n^{-1})$
  - True threshold is $\Theta(n^{-1} \log n)$ because of isolated vertices

The Failure of the First Moment

Subgraph containment

- When is the first moment bound not tight?
  - When the subgraph H is contains a denser subgraph $J \subset H$
  - Appearance of J is a bottleneck for appearance of H
  - Once J appears, it will have many extensions to copies of H
    - $\Rightarrow$ events are far from disjoint
    - $\Rightarrow$ union bound on copies of H is far from tight

Necessary properties

- Goal: determine threshold p_0 of a property $\mathcal{P}$
- Call a monotone property $\mathcal{Q}$ necessary for $\mathcal{P}$ if $\mathrm{G} \in \mathcal{P} \Rightarrow G \in \mathcal{Q}$
  - That is, for $\mathcal{P}$ to occur, $\mathcal{Q}$ has to occur first
- $\Rightarrow$ threshold q_0 for $\mathcal{Q}$ satisfies $q_0 \leq p_0$
- Example: containment of densest part J of a subgraph H

The Expectation Threshold

Definition 7.1.1 (Expectation threshold)

Given a monotone graph property $\mathcal{P}$, we define the *expectation threshold of $\mathcal{P}$* as

$$q(\mathcal{P}) \coloneqq \max\{p_1(\mathcal{Q}) : \mathcal{Q} \text{ is necessary for } \mathcal{P}\}.$$

Lower bound

- For every necessary property $\mathcal{Q}$ with threshold q_0, we have $p_1(\mathcal{Q}) \leq q_0 \leq p_0$
- $\Rightarrow q(\mathcal{P}) \leq p_0$

Subgraph containment

- For each subgraph $J \subseteq H$, the property $\mathcal{Q}_J$ of containing J is necessary
- $\Rightarrow p_0 \geq \max\{n^{-v(J)/e(J)} : J \subseteq H\}$
- Second moment gives matching upper bound

The Kahn-Kalai Conjecture

Recall

- $q(\mathcal{P}) \coloneqq \max\{p_1(\mathcal{Q}) : \mathcal{Q} \text{ is necessary for } \mathcal{P}\} \leq p_0$

Hamiltonicity

- Can show $q(\mathcal{P}) = O(n^{-1}) \ll p_0 = \Theta(n^{-1} \log n)$

Conjecture 7.1.2 (Kahn-Kalai, 2007)

There is some universal constant C such that, for any non-trivial monotone property $\mathcal{P}$ and any n, $p_0 \leq Cq(\mathcal{P}) \log n$.

Remarks

- Implies the first-moment is all we need (up to logarithmic factors)
  - Just requires clever application
- Very optimistic conjecture

Computing Expectation Thresholds

Conjecture 7.1.2 (Kahn-Kalai, 2007)

There is some universal constant C such that, for any non-trivial monotone property $\mathcal{P}$ and any n, $p_0 \leq Cq(\mathcal{P}) \log n$.

Computing $q(\mathcal{P})$

- Find the optimal necessary property $\mathcal{Q}$ and identify minimal elements $\mathcal{Q}_0$
- Compute the expected number of members of $\mathcal{Q}_0$ contained in $G(n, p)$

Problem

- How do we find $\mathcal{Q} \in \{0,1\}^{2^{2^{\binom{[n]}{2}}}}$?

Solution

- Look inside $\mathbb{R}_{\geq 0}^{2^{2^{\binom{[n]}{2}}}}$ instead!

Weighted Necessity

Necessary weightings

- Let $\mathcal{G}$ be the set of all graphs on $[n]$
- Let $g: \mathcal{G} \to \mathbb{R}_{\geq 0}$ be a non-negative weighting of all such graphs
- The total weight $w(G)$ of a graph G is the sum of the weights of its subgraphs
  - $w(G) = \sum_{F \subseteq G} g(F)$
- Say g is a necessary weighting for $\mathcal{P}$ if $G \in \mathcal{P} \Rightarrow w(G) \geq 1$

Threshold lower bound

- If g is a necessary weighting, then $\{G(n,p) \in \mathcal{P}\} \subseteq \{w(G(n,p)) \geq 1\}$
- Markov's Inequality $\Rightarrow \mathbb{P}\big(w(G(n,p)) \geq 1\big) \leq \mathbb{E}[w(G(n,p))]$
  - $\mathbb{E}[w(G(n,p))] = \sum_F g(F) p^{e(F)}$
- $\Rightarrow p_0 \geq \max\left\{p : \sum_F g(F) p^{e(F)} \leq \frac{1}{2}\right\} =: q_1(g)$

Fractional Expectation Thresholds

Definition 7.1.3 (Fractional expectation threshold)

Given a non-trivial monotone property $\mathcal{P}$, we define the *fractional expectation threshold* to be

$$q_f(\mathcal{P}) \coloneqq \max \{q_1(g) \colon g \text{ is a necessary weighting for } \mathcal{P}\}.$$

Bounds

- $q(\mathcal{P}) \leq q_f(\mathcal{P}) \leq p_0$
  - $\mathcal{Q}$ is necessary for $\mathcal{P} \Rightarrow 1_{\mathcal{Q}_0}$ necessary for $\mathcal{P}$, and $q_1(\mathcal{Q}) = q_1(1_{\mathcal{Q}_0})$

Conjecture 7.1.4 (Talagrand, 2010)

There is some universal constant C such that, for any non-trivial monotone property $\mathcal{P}$ and any n, $p_0 \leq C q_f(\mathcal{P}) \log n$.

Comparing Conjectures

Conjecture 7.1.2 (Kahn-Kalai, 2007)

There is some universal constant C such that, for any non-trivial monotone property $\mathcal{P}$ and any n, $p_0 \leq Cq(\mathcal{P}) \log n$.

Conjecture 7.1.4 (Talagrand, 2010)

There is some universal constant C such that, for any non-trivial monotone property $\mathcal{P}$ and any n, $p_0 \leq Cq_f(\mathcal{P}) \log n$.

- Since $q(\mathcal{P}) \leq q_f(\mathcal{P})$, Kahn-Kalai ⇒ Talagrand

Conjecture 7.1.5 (Talagrand, 2010)

There is some universal constant $c > 0$ such that, for any non-trivial monotone property $\mathcal{P}$ and any n, $q(\mathcal{P}) \geq cq_f(\mathcal{P})$.

The Big Breakthrough

Theorem 7.1.6 (Frankston-Kahn-Narayanan-Park, 2019+)

There is a universal constant C such that, for any non-trivial monotone property $\mathcal{P}$ and any n, $p_0 \leq C q_f(\mathcal{P}) \log n$.

Moreover, the $\log n$ can be replaced by $\log l(\mathcal{P})$, where $l(\mathcal{P})$ is the maximum size of a minimal graph in $\mathcal{P}$.

Remarks

- $\Rightarrow q_f(\mathcal{P}) \leq p_0 \leq C q_f(\mathcal{P}) \log l(\mathcal{P})$
  - First moment is almost sharp
- Trivial bound: $l(\mathcal{P}) \leq \binom{n}{2}$
  - $\Rightarrow \log l(\mathcal{P}) = O(\log n)$
- Extends to more general settings
  - Hypergraphs, integers, etc.

Any questions?

§2 Applications

Chapter 7: Expectation Thresholds

The Probabilistic Method

Applying the Theorem

Theorem 7.1.6 (Frankston-Kahn-Narayanan-Park, 2019+)

There is a universal constant C such that, for any non-trivial monotone property $\mathcal{P}$ and any n, $p_0 \leq C q_f(\mathcal{P}) \log l(\mathcal{P})$, where $l(\mathcal{P})$ is the maximum size of a minimal graph in $\mathcal{P}$.

Applications

- Need to estimate $q_f(\mathcal{P})$
  - Gives us the threshold, up to a logarithmic factor
- $q_f(\mathcal{P})$: maximum over all necessary weightings
  - Much larger search space than testing necessary properties!
  - Continuous setting allows for linear programming
  - Easier to obtain some bounds

Bounding the Fractional Expectation Threshold

Proposition 7.2.1

Let $\mathcal{P}$ be a non-trivial monotone property, and let $\mathcal{P}_0 \subseteq \mathcal{P}$ be arbitrary. Then

$$q_f(\mathcal{P}) \leq \max_{\emptyset \neq F \subseteq K_n} \left(\frac{|\mathcal{P}_0(F)|}{|\mathcal{P}_0|} \right)^{\frac{1}{e(F)}},$$

where $\mathcal{P}_0(F) = \{G \in \mathcal{P}_0 : F \subseteq G\}$.

Remarks

- Typically take $\mathcal{P}_0$ to be minimal graphs in $\mathcal{P}$
  - Have the freedom to restrict to a well-behaved subclass of graphs
- For fractional expectation threshold to be low, want $\mathcal{P}_0$ to be spread out

Proving the Bound

Proof

- Let g be any necessary weighting for $\mathcal{P}$
  - $\Rightarrow \sum_{F \subseteq G} g(F) \geq 1$ for all $G \in \mathcal{P}$
- $\Rightarrow |\mathcal{P}_0| = \sum_{G \in \mathcal{P}_0} 1 \leq \sum_{G \in \mathcal{P}_0} \sum_{F \subseteq G} g(F)$

$$= \sum_F \sum_{G \supseteq F, G \in \mathcal{P}_0} g(F)$$

$$= \sum_F g(F) |\mathcal{P}_0(F)|$$

$$\Rightarrow 1 \leq \sum_F g(F) \left(\frac{|\mathcal{P}_0(F)|}{|\mathcal{P}_0|} \right)$$

- If $p \geq \max_{F \subseteq K_n} \left(\frac{|\mathcal{P}_0(F)|}{|\mathcal{P}_0|} \right)^{\frac{1}{e(F)}}$, then

$$\mathbb{E}[w(G(n,p))] = \sum_F g(F) p^{e(F)} \geq \sum_F g(F) \left(\frac{|\mathcal{P}_0(F)|}{|\mathcal{P}_0|} \right) \geq 1$$

- $\Rightarrow p \geq q_1(g)$
- g an arbitrary necessary weighting $\Rightarrow p \geq q_f(\mathcal{P})$ ■

Subgraph Containment

Corollary 7.2.2

Let H be a fixed graph, and let $\mathcal{P}_H$ be the property of containing H as a subgraph. The threshold for $\mathcal{P}_H$ satisfies $p_0 = \Theta\left(\mathrm{n}^{-1/\mathrm{m(H)}}\right)$, where $m(H) = \max\left\{\frac{e(J)}{v(J)} : J \subseteq H, v(J) \geq 1\right\}$.

Proof

- Lower bound: first moment
- Let $\mathcal{P}_0$ be the minimal graphs in $\mathcal{P}_H$, i.e. copies of H in K_n
  - $\Rightarrow |\mathcal{P}_0| \geq \binom{n}{v(H)} = \Theta\left(n^{v(H)}\right)$ and $l(\mathcal{P}) = O(1)$
- For any F, if $F \nsubseteq H$, then $\mathcal{P}_0(F) = \emptyset$
- If $F \subseteq H$, then $|\mathcal{P}_0(F)| \leq v(H)^{v(F)} n^{v(H)-v(F)} = \Theta\left(n^{v(H)-v(F)}\right)$
- $\Rightarrow q_f(\mathcal{P}) \leq \max_F \left(\frac{|\mathcal{P}_0(F)|}{|\mathcal{P}_0|}\right)^{1/e(F)} \sim \max_F n^{\frac{-v(F)}{e(F)}} = n^{\frac{-1}{m(H)}}$ ■

That Was Easy

Corollary 7.2.2

Let H be a fixed graph, and let $\mathcal{P}_H$ be the property of containing H as a subgraph. The threshold for $\mathcal{P}_H$ satisfies $p_0 = \Theta\left(\mathrm{n}^{-1/\mathrm{m(H)}}\right)$, where $m(H) = \max\left\{\frac{e(J)}{v(J)} : J \subseteq H, v(J) \geq 1\right\}$.

Remarks

- Agrees with earlier result from Chapter 3
  - No need to mess about with the second moment!
- Proof generalises directly to r-uniform hypergraphs
  - Appearance threshold is again $n^{-1/m(H)}$

Spanning Hypergraphs

Interesting applications

- Typically concern spanning hypergraphs $H \subseteq K_n^{(r)}$

Parameters

- $d^*(v) = |\{w \neq v : d_H(v,w) \geq 1\}|$: size of neighbourhood of a vertex
- $\Delta^*(H) = \max_{v \in V} d^*(v)$
- $m_1(H) = \max\left\{\frac{e(J)}{v(J)-1} : J \subseteq H, v(J) \geq 2\right\}$

Lemma 7.2.3

Let H be a spanning r-uniform hypergraph, and let $\mathcal{P}_H$ be the property of containing a copy of H. Then

$$q_f(\mathcal{P}_H) \leq e \cdot \left(\frac{e \cdot \Delta^*(H)}{n}\right)^{\frac{1}{m_1(H)}}.$$

Hamiltonicity

Corollary 7.2.4

The threshold p_0 for $G(n,p)$ to be Hamiltonian satisfies $p_0 = O\left(\frac{\log n}{n}\right)$.

Proof

- Theorem 7.1.6 $\Rightarrow p_0 \le C q_f(\mathcal{P}_H) \log n$, where $\mathcal{P}_H$ is the property for containing a Hamiltonian cycle $H = C_n$
- $\Delta^*(H) = 2$
- $m_1(H) = \frac{n}{n-1}$
  - If $J = H$, then $\frac{e(J)}{v(J)-1} = \frac{n}{n-1}$
  - Otherwise J is a path-forest, and $\frac{e(J)}{v(J)-1} \le 1$
- Lemma 7.2.3 $\Rightarrow q_f(\mathcal{P}_H) = O\left(\left(\frac{2e}{n}\right)^{\frac{n-1}{n}}\right) = O(n^{-1})$ ∎

Hamiltonicity: A Review

Lower bound

- We know $p_0 = \Omega\left(\frac{\log n}{n}\right)$, as we have isolated vertices below this threshold

Upper bound

- Previously proved threshold using Pósa rotations
- This proof is *much* simpler

Sharp thresholds

- This approach only gives the order of magnitude of the threshold
- Sharper techniques with rotations show a sharp threshold around $p \sim \frac{\log n}{n}$

Sharpness of Theorem 7.1.6

- Hamiltonicity shows that the logarithmic factor can be necessary

Perfect Matchings

Corollary 7.2.5

When $r|n$, the threshold for the random r-uniform hypergraph $H^{(r)}(n,p)$ to contain a perfect matching is $p_0 = \Theta(n^{1-r}\log n)$.

Proof

- Lower bound: if $p = o(n^{1-r}\log n)$, then we have isolated vertices
- Upper bound: let H be a perfect matching
  - $\Delta^*(H) = r - 1$
  - $m_1(H) = \frac{1}{r-1}$
- Lemma 7.2.3 $\Rightarrow q_f(H) = O\left(\left(\frac{(r-1)e}{n}\right)^{r-1}\right) = O(n^{1-r})$
- Theorem 7.1.6 $\Rightarrow p_0 = O(n^{1-r}\log n)$ ■

Shamir’s Problem – A Retrospective

History

- Threshold for perfect matching first asked by Shamir in 1979
- Appeared in Erdős paper: “*On the combinatorial problems which I would most like to see solved*”
  - Said that it was not at all obvious what the threshold should be
- Eventually resolved by Johansson, Kahn and Vu in 2008
  - Proof is notoriously difficult!

Necessity of other techniques

- Lower bound used existence of isolated vertices
  - Requires a second-moment calculation
- First-moment alone leaves the logarithmic gap
  - Necessary for graph factors!

Spanning Trees

Corollary 7.2.6

Let H be a given spanning tree with maximum degree Δ and let $\mathcal{P}_H$ be the property of containing a copy of H. The threshold for $G(n,p)$ to have this property satisfies $p_0(\mathcal{P}_H) = \Theta\left(\frac{\log n}{n}\right)$.

Proof

- Lower bound: if $p = o\left(\frac{\log n}{n}\right)$, then $G(n,p)$ has isolated vertices w.h.p.
- Upper bound: for H we have
  - $\Delta^* = \Delta$
  - $m_1(H) = 1$ (every $J \subseteq H$ is a forest $\Rightarrow e(J) \leq v(J) - 1$)
- Lemma 7.2.3 $\Rightarrow q_f(\mathcal{P}_H) \leq e\left(\frac{e\Delta}{n}\right) = O\left(\frac{1}{n}\right)$
- Theorem 7.1.6 $\Rightarrow p_0(\mathcal{P}_H) = O\left(\frac{\log n}{n}\right)$ ■

A History of Trees

Conjecture

- Kahn (1990's) conjectured threshold for spanning tree appearance
- Proven for trees with certain properties, e.g.:
  - Trees with $\Omega(n)$ leaves
  - Trees with bare paths of length $\Omega(n)$
  - Combs (a path of $\frac{n}{k}$ paths of length k) with
    - $k = O(\log n)$
    - $k = \Omega(\log n)$
  - Typical trees

Resolution

- Proven by Montgomery (2019)
  - Development of powerful absorption technique
  - Shows more: if $p \geq \frac{C_\Delta \log n}{n}$, $G(n,p)$ contains *every* spanning tree H with $\Delta(H) \leq \Delta$

Proving the Lemma

Lemma 7.2.3

Let H be a spanning r-uniform hypergraph, and let $\mathcal{P}_H$ be the property of containing a copy of H. Then

$$q_f(\mathcal{P}_H) \leq e \cdot \left(\frac{e \cdot \Delta^*(H)}{n}\right)^{\frac{1}{m_1(H)}}.$$

Proof

- Proposition 7.2.1 $\Rightarrow$ for any $\mathcal{P}_0 \subseteq \mathcal{P}_H$, we have
  - $q_f(\mathcal{P}_H) \leq \max\left\{\left(\frac{|\mathcal{P}_0(F)|}{|\mathcal{P}_0|}\right)^{\frac{1}{e(F)}} : \emptyset \neq F \subseteq K_n\right\}$
  - where $\mathcal{P}_0(F) = \{G \in \mathcal{P}_0 : F \subseteq G\}$
- We take $\mathcal{P}_0$ to be the minimal members of $\mathcal{P}_H \leftrightarrow$ copies of H in K_n
  - $\Rightarrow \mathcal{P}_0(F) = \emptyset$ unless $F \subseteq H$

Continuing to Prove the Lemma

Goal

- Bound maximum of $\left(\frac{|\mathcal{P}_0(F)|}{|\mathcal{P}_0|}\right)^{\frac{1}{e(F)}}$ by $e\left(\frac{e\cdot\Delta^*(H)}{n}\right)^{\frac{1}{m_1(H)}}$

Count *labelled* copies of H

- $\mathcal{P}_0 = n!$ bijections from $V\big(G(n,p)\big)$ to $V(H)$
- $\mathcal{P}_0(F) =$ bijections sending edges of F to edges of H

Structure of F

- Let F have t connected components with at least one edge
  - Label components $F_1, F_2, \ldots, F_t$
  - Let $v_i = v(F_i)$ and $e_i = e(F_i)$
  - Let $v = \sum_i v_i$ be the number of non-isolated vertices in F

Continuing to Continue to Prove the Lemma

Recall

- $\mathcal{P}_0(F)$: bijections from $V\big(G(n,p)\big)$ to $V(H)$ sending edges of F to edges of H

Mapping a component

- Send the first vertex of F_i to H
  - At most n options
- Since component is connected, can order vertices so that each later vertex has an earlier neighbour
  - Image should also be a neighbour in H of earlier image
  - At most Δ^* options
- $\Rightarrow$ at most $n(\Delta^*)^{v_i - 1}$ choices of image of F_i

Total count

- $\Rightarrow |\mathcal{P}_0(F)| \leq n^t(\Delta^*)^{v-t}(n-v)!$

Finishing the Proof of the Lemma

Bounding the ratio

- $\frac{|\mathcal{P}_0(F)|}{|\mathcal{P}_0|} = \frac{n^t(\Delta^*)^{v-t}(n-v)!}{n!} \leq e^t \cdot \left(\frac{e \cdot \Delta^*}{n}\right)^{v-t}$
  - since $\frac{(n-v)!}{n!} \leq \left(\frac{e}{n}\right)^v$
- $\Rightarrow \left(\frac{|\mathcal{P}_0(F)|}{|\mathcal{P}_0|}\right)^{\frac{1}{e(F)}} \leq e^{t/e(F)} \cdot \left(\frac{\mathrm{e} \cdot \Delta^*}{n}\right)^{(v-t)/e(F)}$

Bounding the exponents

- Since each component F_i contains an edge, $t \leq e(F)$
- $\frac{v-t}{e(F)} = \frac{\sum_i (v_i - 1)}{\sum_i e_i} \geq \min\left\{\frac{v_i - 1}{e_i} : i \in [t]\right\} = \left(\max\left\{\frac{e_i}{v_i - 1} : i \in [t]\right\}\right)^{-1} \geq m_1(H)^{-1}$

Finishing the calculation

- $\Rightarrow q_f(\mathcal{P}_H) \leq \max\left\{\left(\frac{|\mathcal{P}_0(F)|}{|\mathcal{P}_0|}\right)^{\frac{1}{e(F)}} : \emptyset \neq F \subseteq K_n\right\} \leq e \cdot \left(\frac{e \cdot \Delta^*}{n}\right)^{\frac{1}{m_1(H)}}$ ■

Any questions?