Counting Designs

Paper by Peter Keevash

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Introduction

- A simple (n, q, r, λ)-design is a family $\mathcal{F} \in \binom{[n]}{q}$ such that every $A \in \binom{[n]}{r}$ is contained in exactly λ members of $\mathcal{F}$.
- **Divisibility conditions**

$$\binom{q-i}{r-i} \text{ divides } \lambda\binom{n-i}{r-i}, \quad 0 \leq i \leq r-1. \qquad (1)$$

Theorem (Keevash)

For fixed q, r, and λ, there exists $n_0(q, r, \lambda)$ such that if $n > n_0(q, r, \lambda)$ satisfies the divisibility conditions (1) then an (n, q, r, λ)-design exists.

- Proving Keevash's theorem for $(n, 3, 2, 1)$-designs.

When does a graph have triangle decomposition?

- When does a graph G have triangle decompositions? i.e, G is edge-disjoint of triangles.
- Assume that G has a triangle decomposition.
  - i) 3 divides $e(G)$.
  - ii) $\deg(v)$ is even for all $v \in V(G)$.
- **Definition.** G is *tridivisible* if (i) and (ii) hold.
- Does every tridivisible graph have a triangle decomposition?
- Answer: No, e.g C_6.

(c,h)-typical graph

- G: graph on n vertices. $G(v)$: set of neighbors of v.
- *Density* of G is $d(G) = e(G)/\binom{n}{2}$.
- G is **(c,h)-typical** if for any $S \subseteq V(G)$ with $|S| \leq h$, then

$$|\cap_{x \in S} G(x)| = |G(S)| = (1 \pm |S|c)d(G)^{|S|}n.$$

- For example, G is $(c,2)$-typical (or c-typical) if
  - i) $|G(v)| = (1 \pm c)d(G)n, \forall v \in V(G)$,
  - ii) $|G(u) \cap G(v)| = (1 \pm 2c)d(G)^2 n, \forall u, v \in V(G), u \neq v$.
- Example. K_n is $1/n$-typical. We have $d(K_n) = 1$
  - i) $|G(v)| = n - 1 = n \pm 1 = (1 \pm 1/n)n (= (1 \pm c)d(G)n)$,
  - ii) $|G(u) \cap G(v)| = n - 2 = (n \pm 2)(= (1 \pm 2c)d(G)^2 n)$.

Keevash's main Theorem

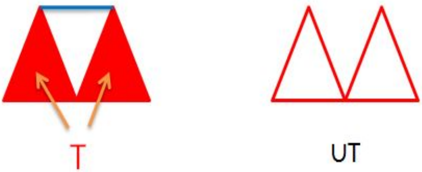
Theorem (Keevash, 2015)

There exists $0 < c_0 < 1$ *and* $n_0 \in \mathbb{N}$ *so that if* $n \geq n_0$ *and* G *is a* $(c, 16)$*-typical tridivisible graph on* n *vertices with* $d(G) > n^{-10^{-7}}$ *and* $c < c_0 d(G)^{10^6}$ *then* G *has a triangle decomposition.*

- The theorem is also true for $(c, 2)$-typical graphs.
- Density decays polynomially with n.
- Theorem holds for large n and small c.

Strategy for proof

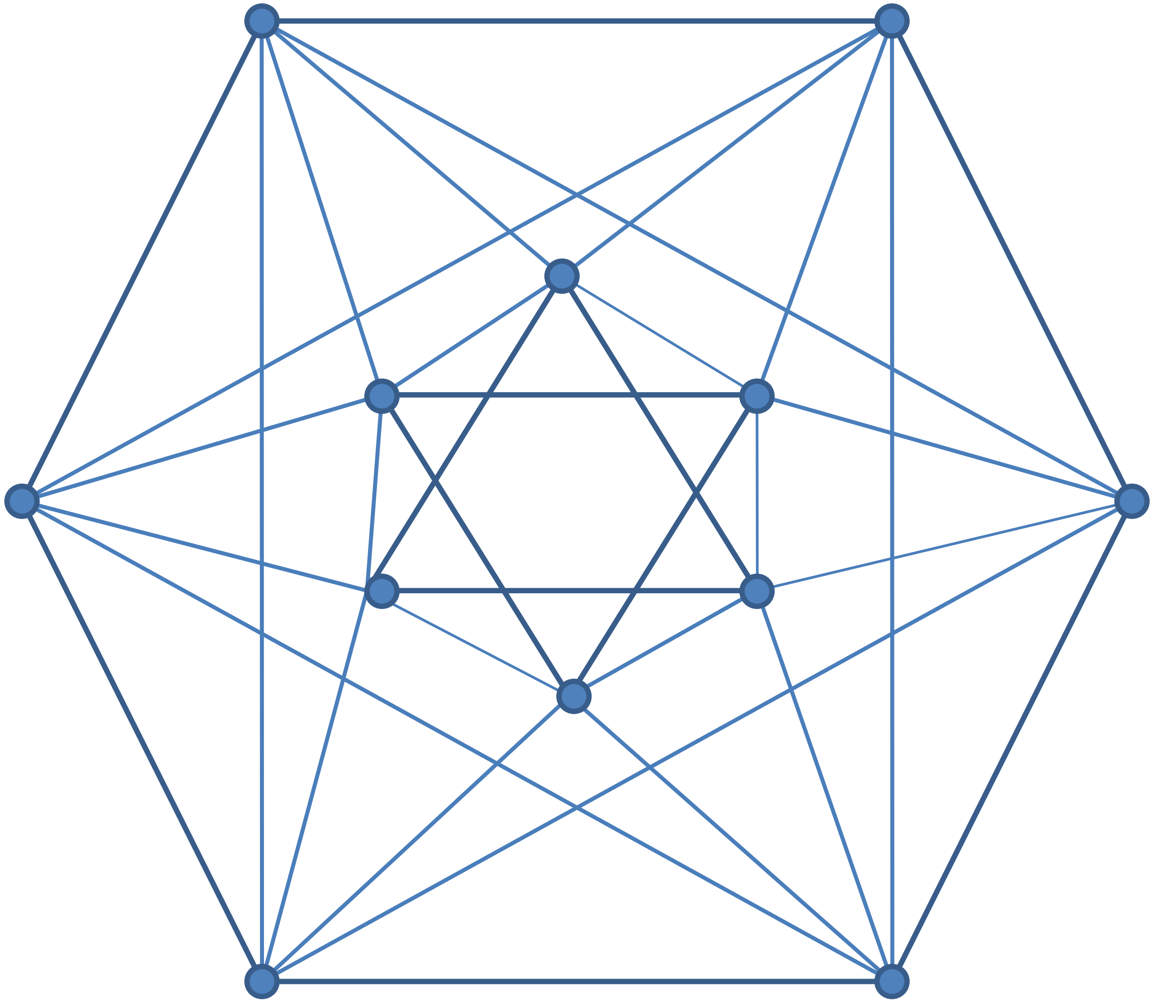
- Use Randomized Algebraic Construction method.
- Combination of algebraic structure with the probabilistic constructions.
- The proof has 5 stages (in 4 lectures).
- **Notation.** Let $K_3(G)$ denote the set of triangles in G.
- Let T be set of edge-disjoint triangle of G. We write $\cup T$, the union of all edges in triangles in T.



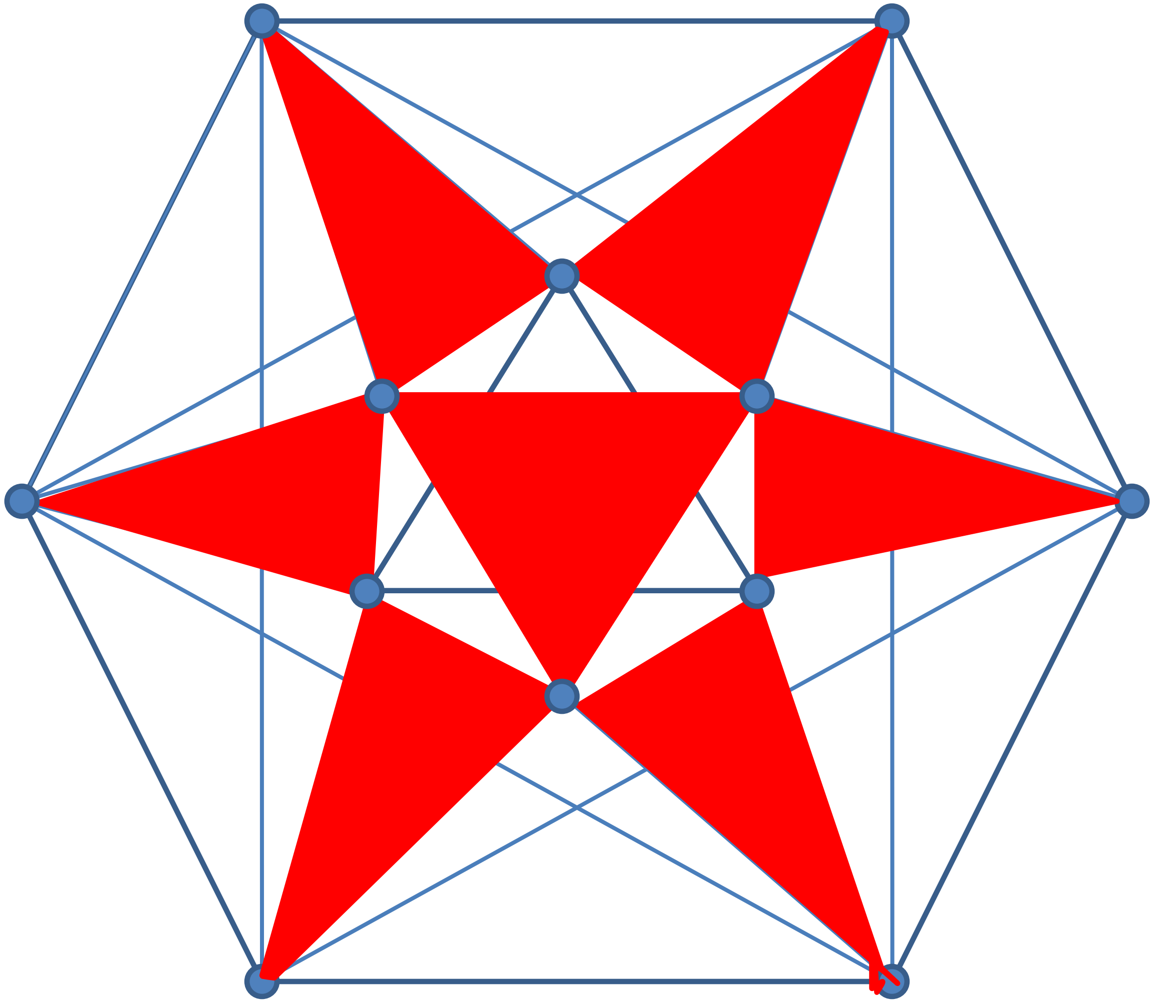
5 stages for proof

- $J \subseteq G$ is c-bounded if $|J(v)| < cn$, for every $v \in V(G)$, where $J(v) = \{u \in V(G) : uv \in J\}$ is the neighbourhood of v in J.

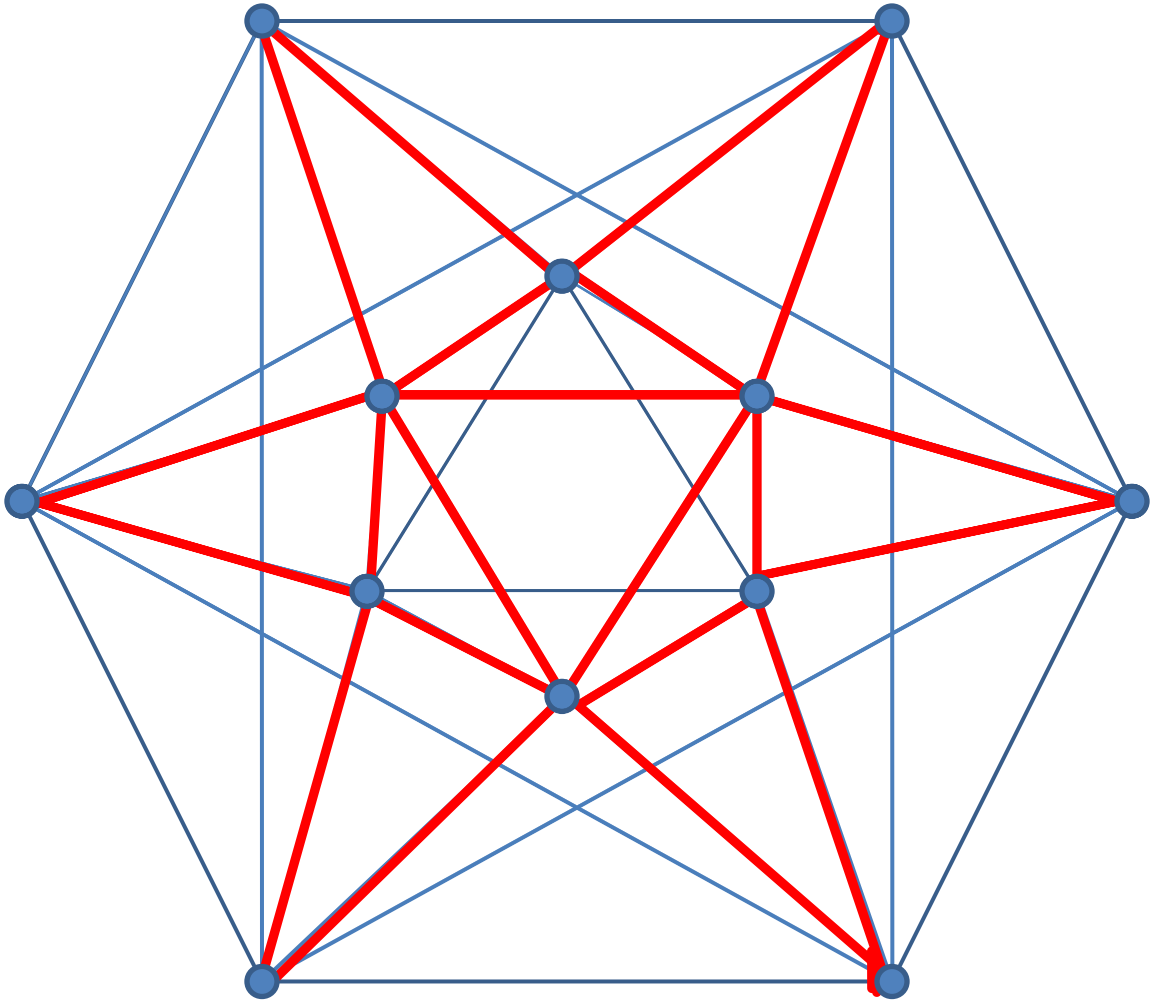
Find triangle decomposition



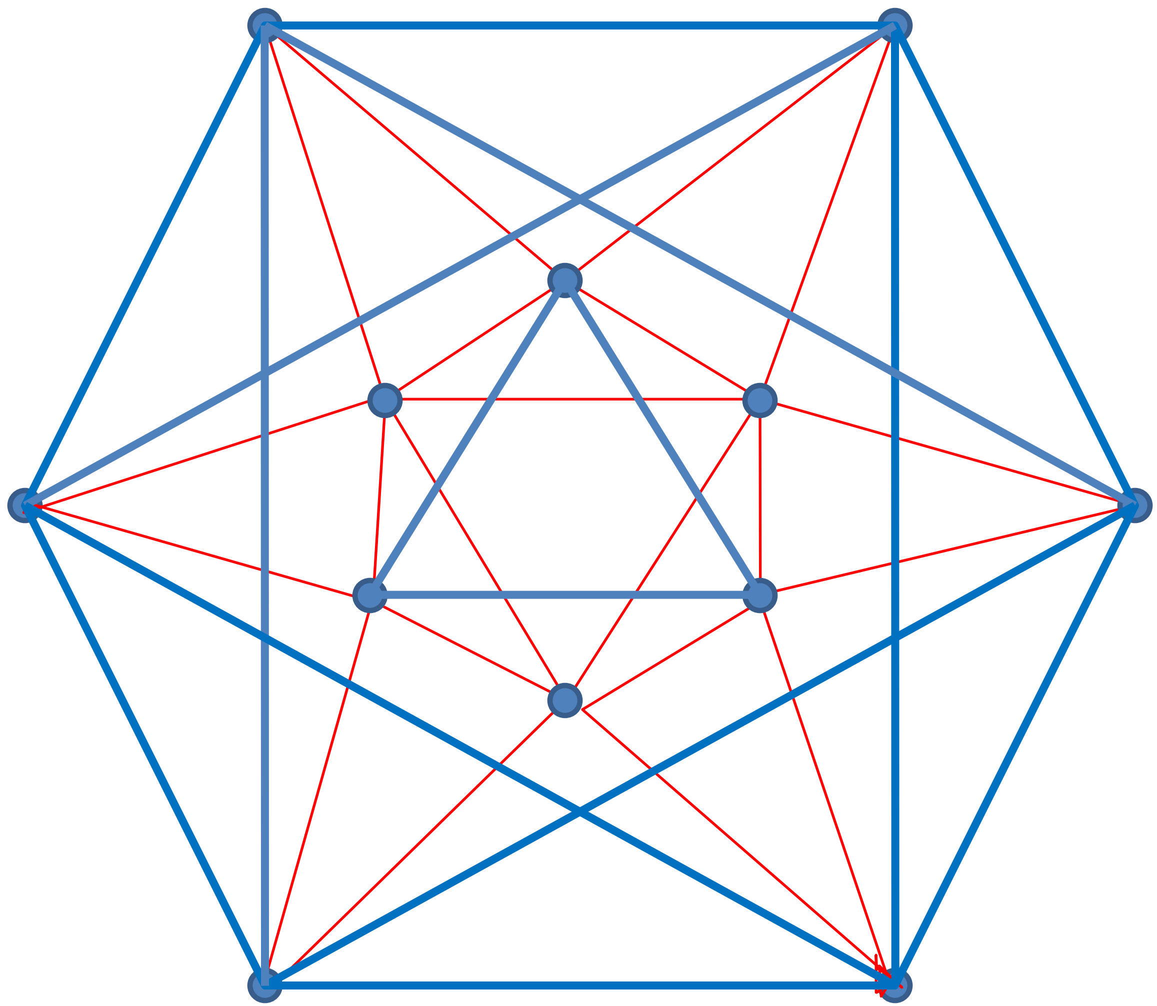
1. **Template.** Find a SET (set of edge-disjoint triangle) T in $K_3(G)$ via an algebraic construction.



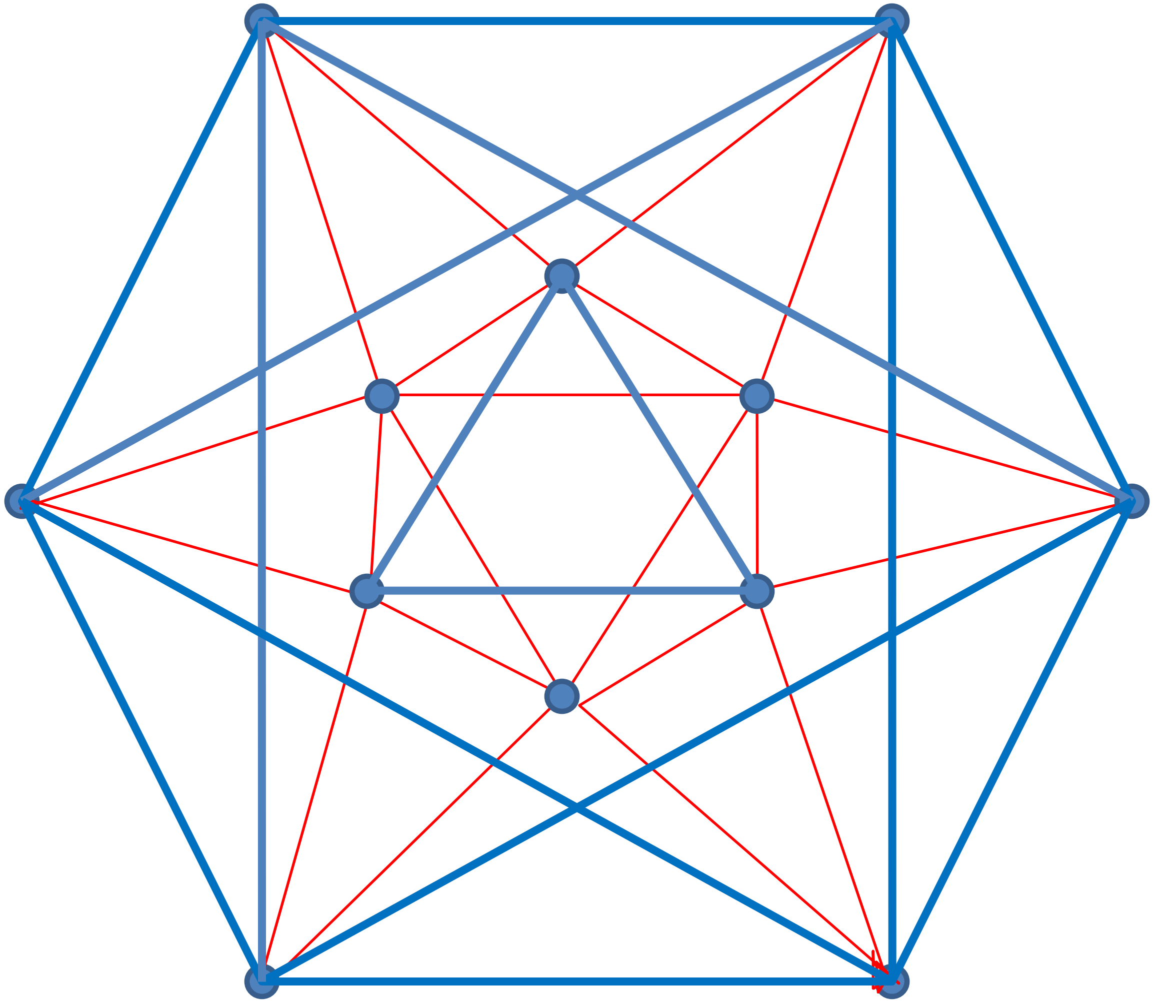
$G^* = \cup T \subseteq G.$



G\G*



2. Nibble: Find a SET **N** in G\G*.



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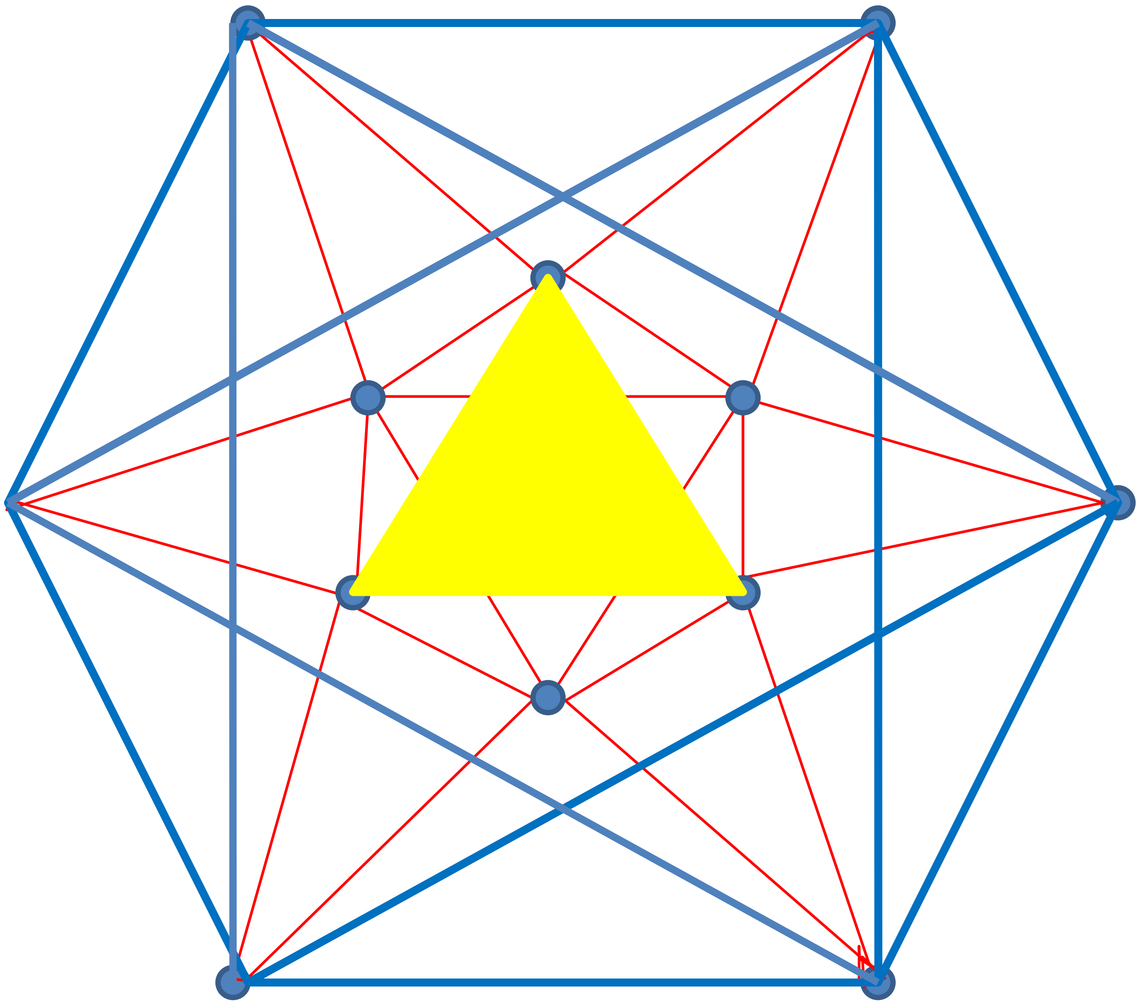

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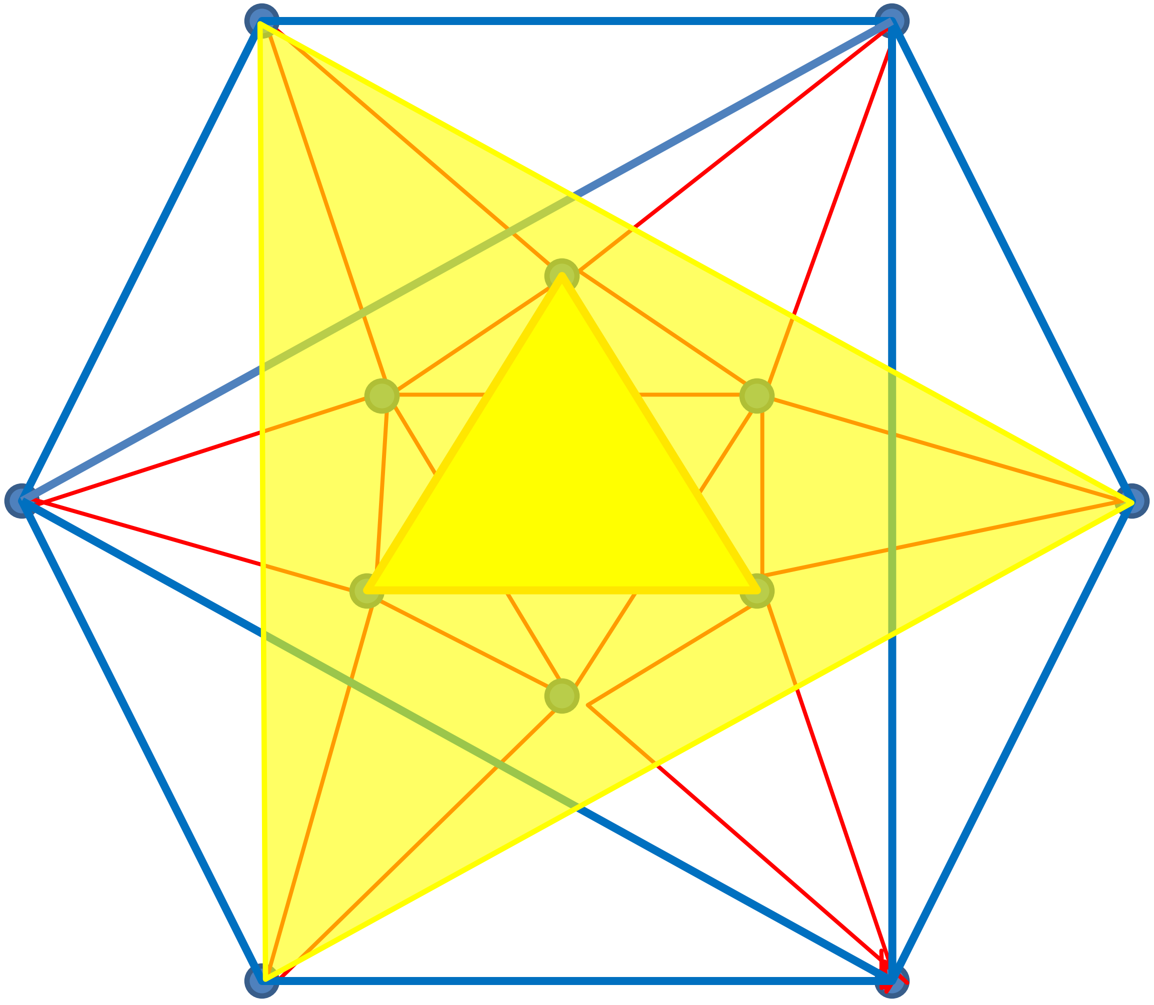
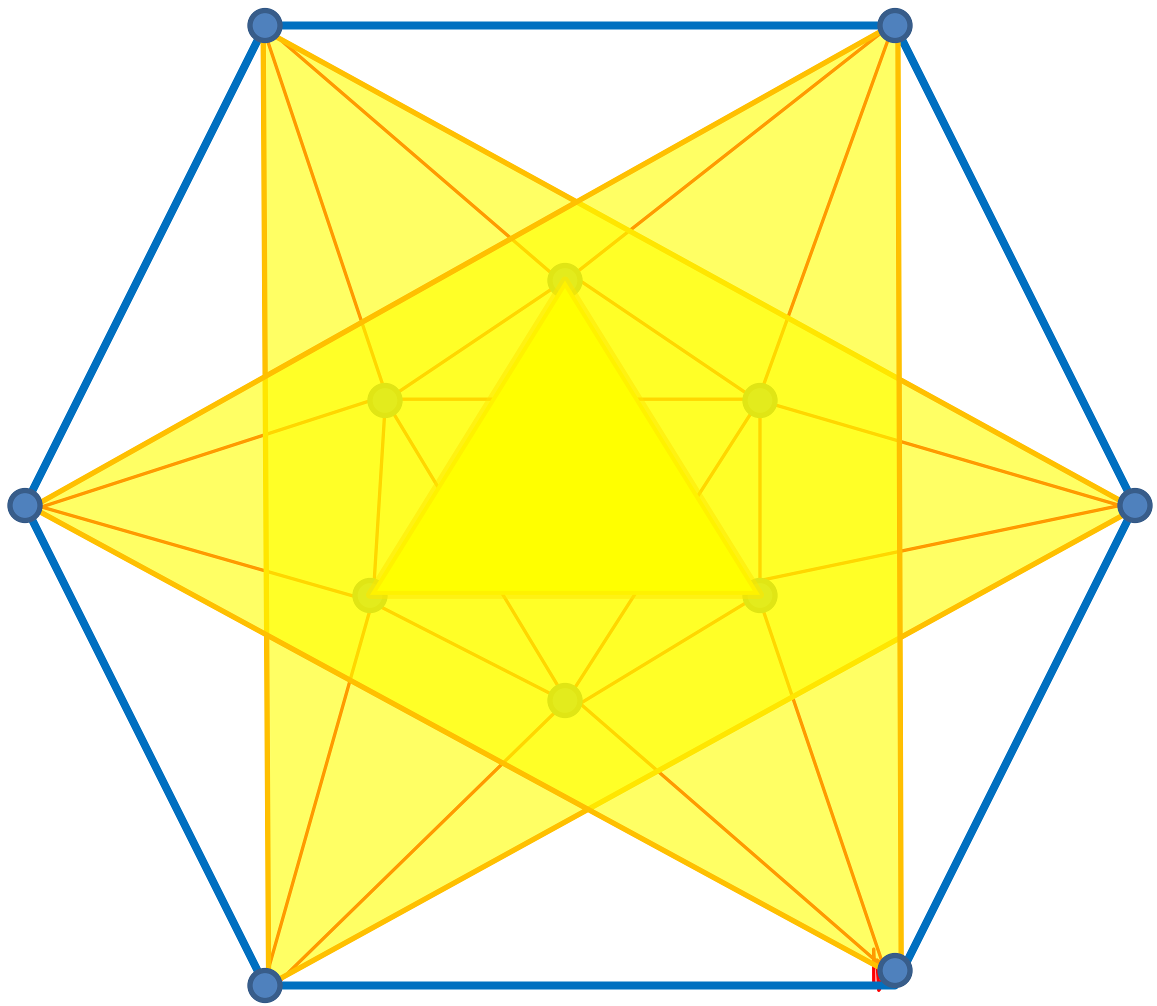
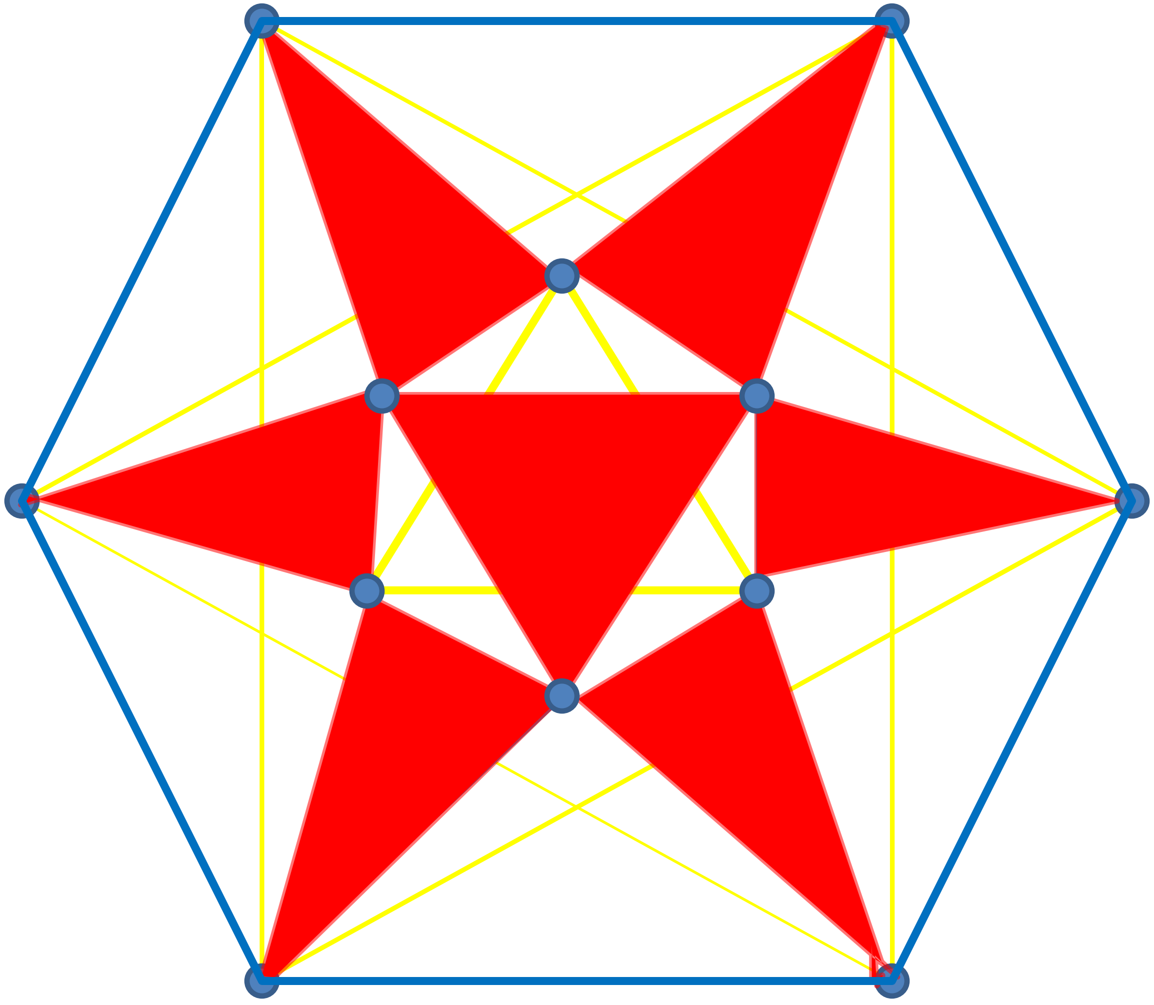


2. Nibble: We want the leave L:= (G\G*)\∪N is c_1-bounded.



3. **Cover**: Given L, cover each edge in L by triangle using two edges in G* s.t new triangle is edge-disjoint from previous choices.



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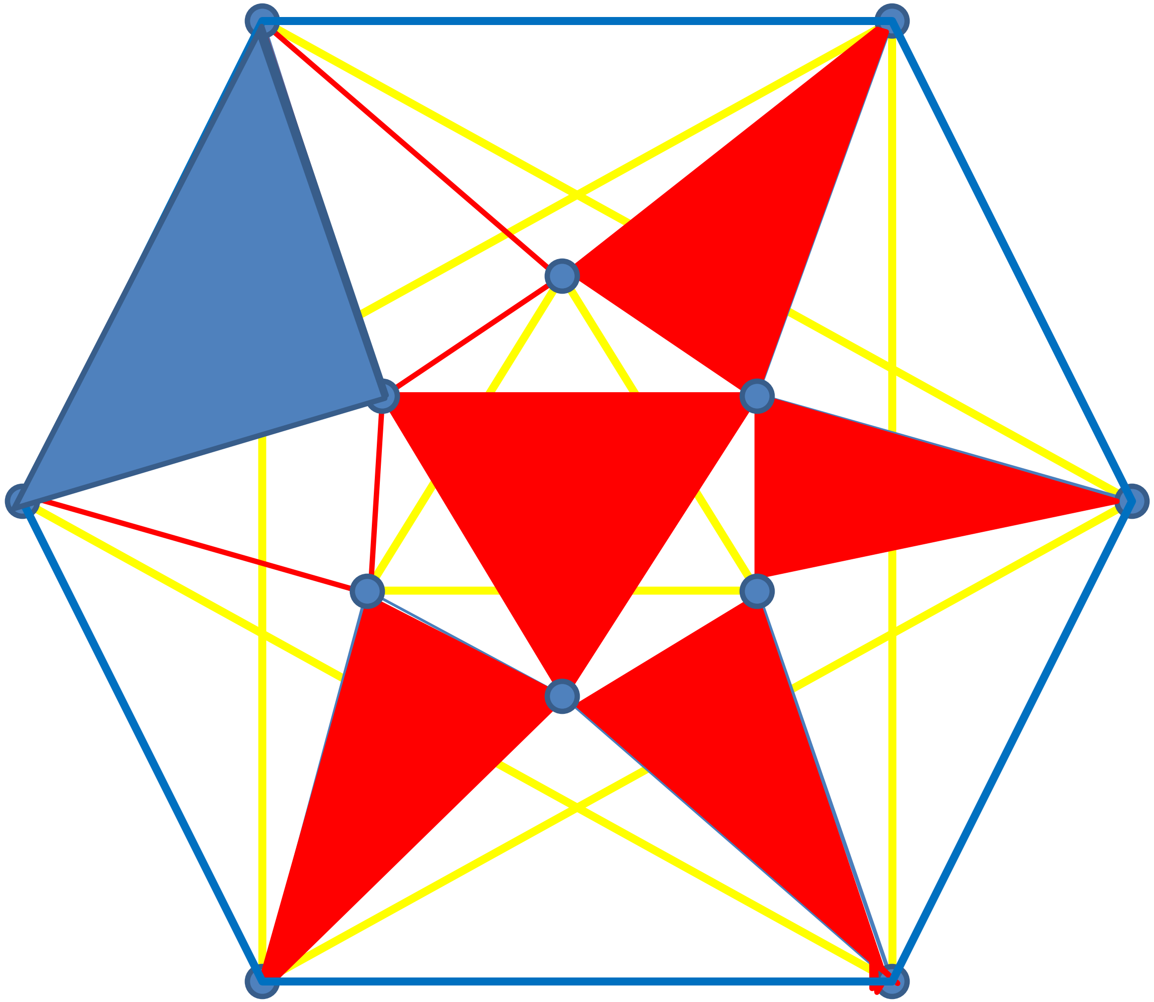

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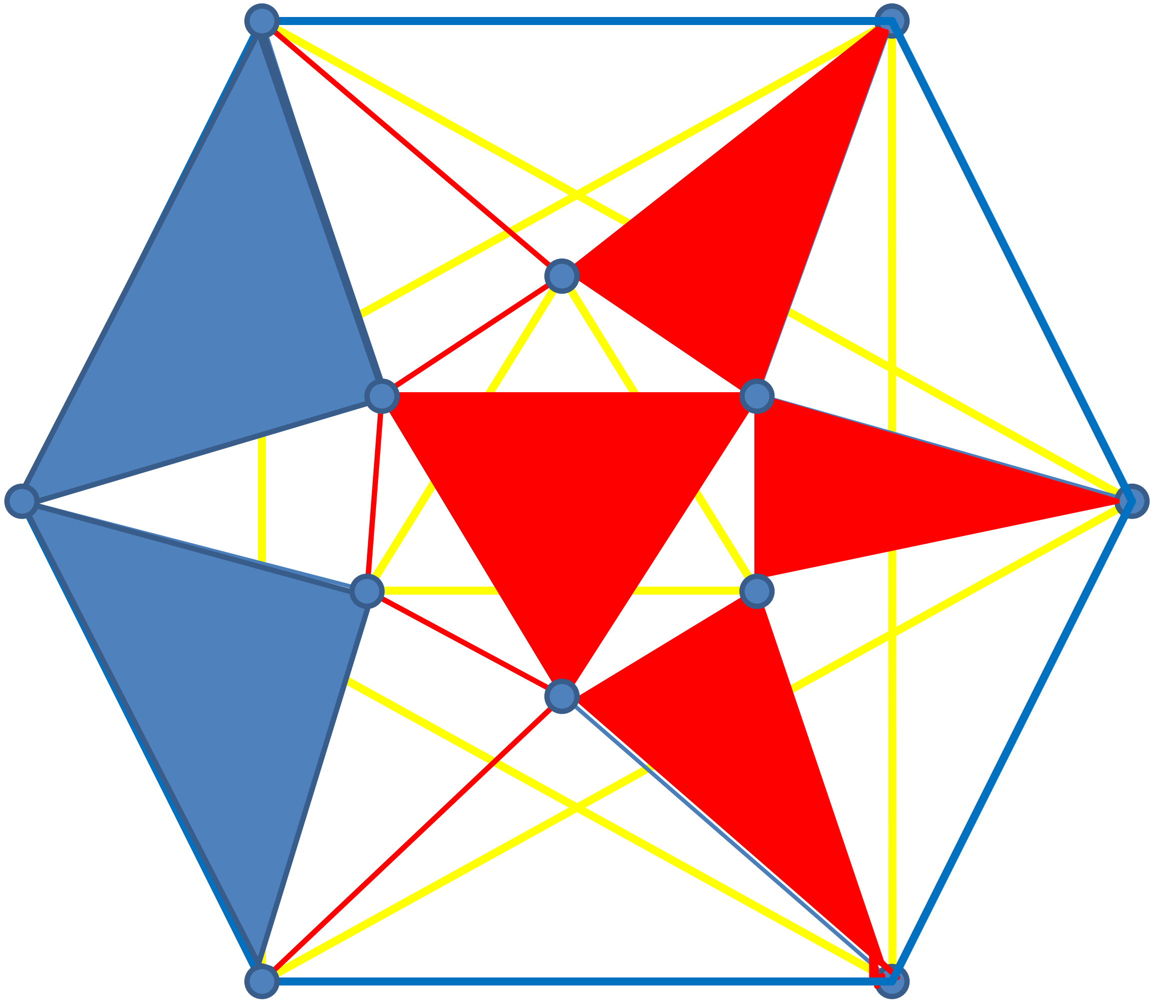

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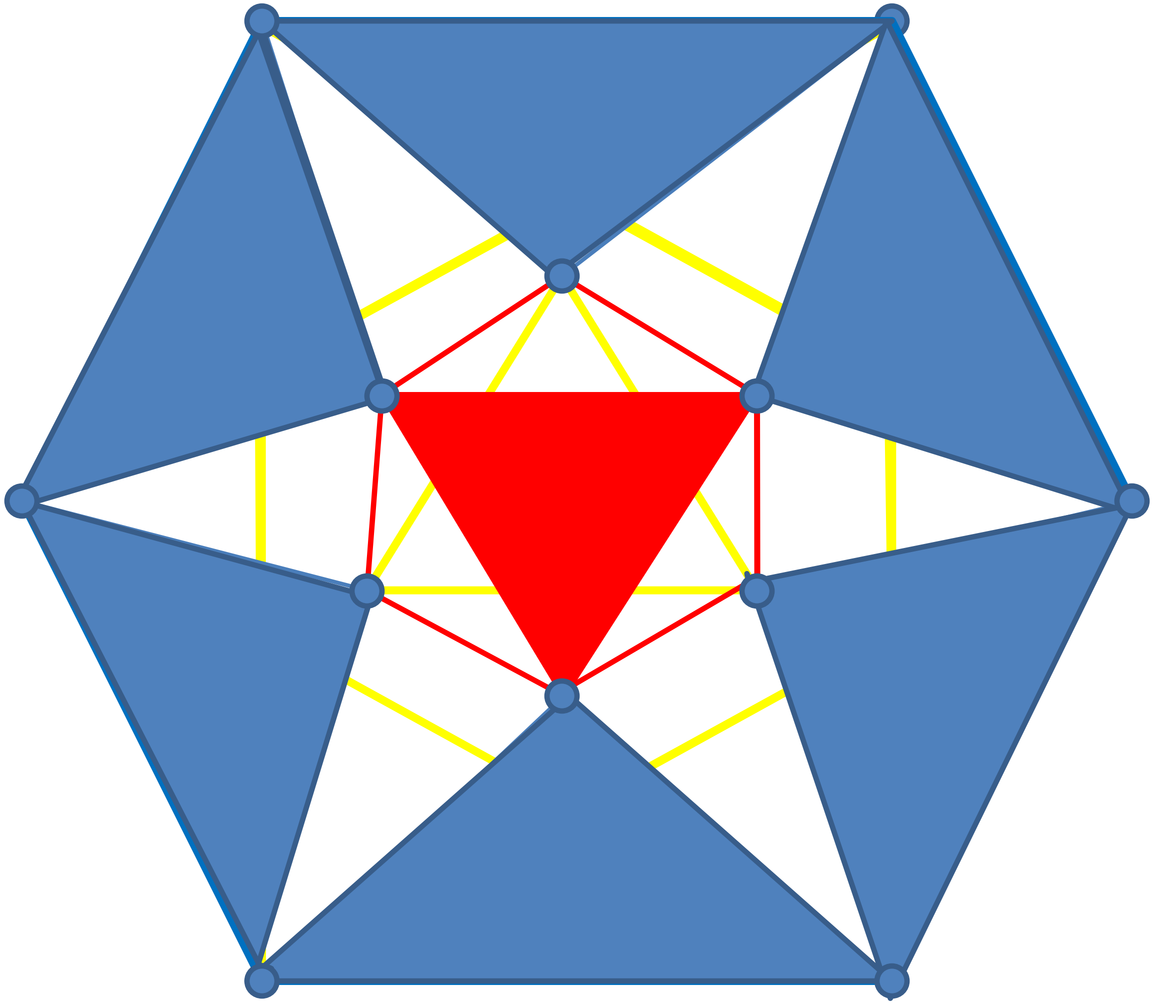
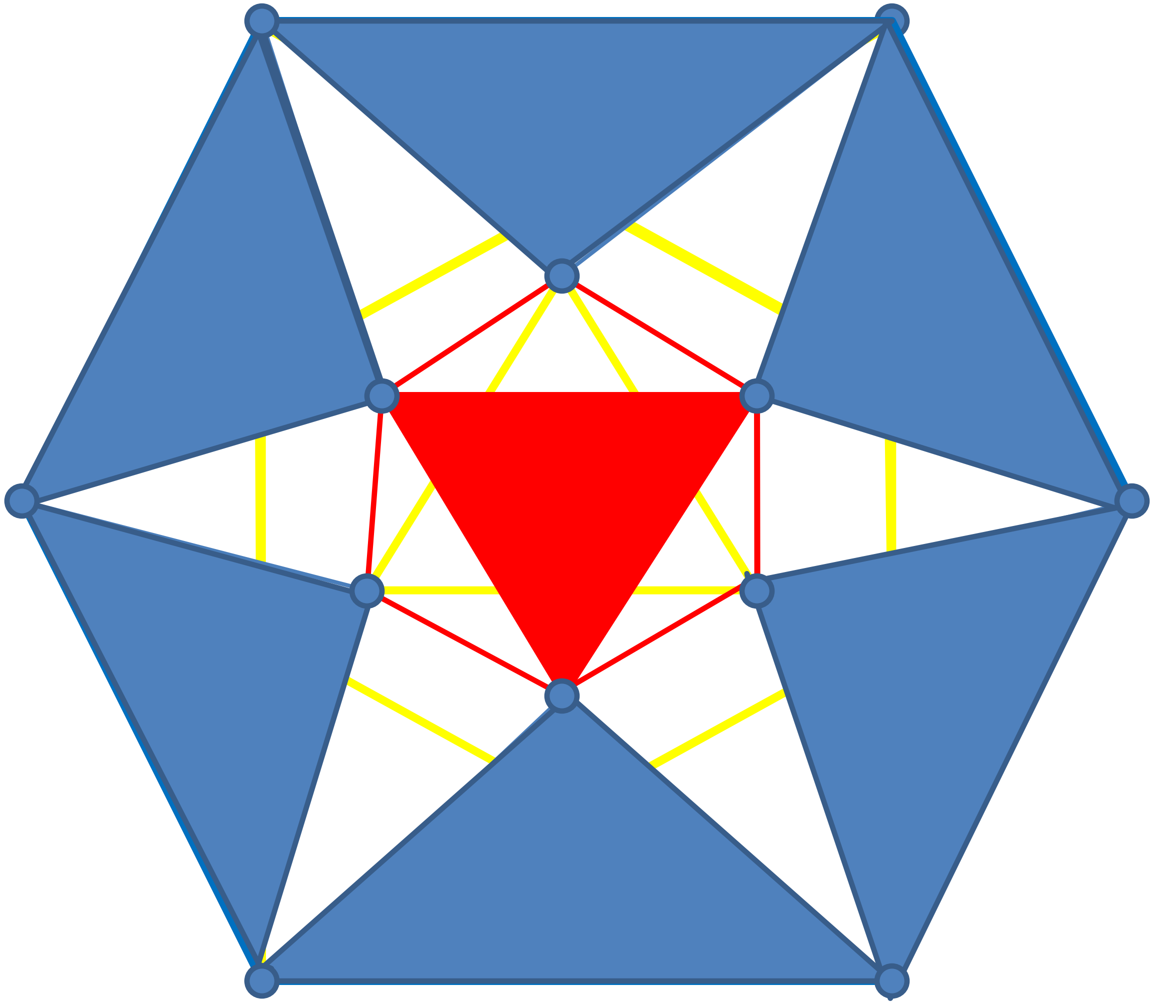


4. & 5. **Hole & Completion.** Switch triangles in T to get a triangle decomposition.



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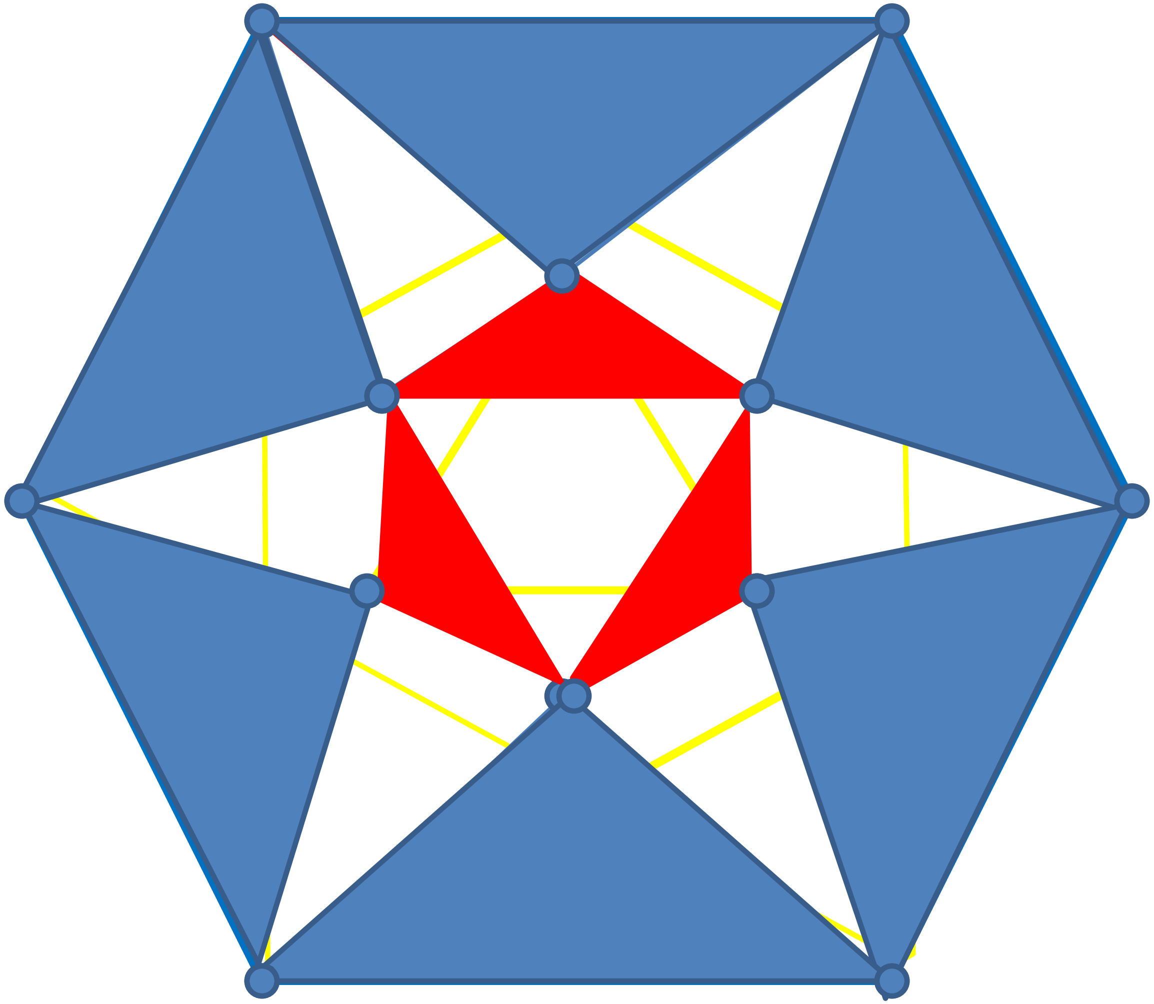

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