Counting Designs — Nibble, Cover

Brent Moran

FU DMIII Seminar WS 2016

Quick Review — Definitions

Definition

Recall that $d(G)$ denotes the $\boxed{\text{density}}$ of the graph $d(G) = \frac{|E|}{\binom{n}{2}}$.

Quick Review — Definitions

Definition

Recall that $d(G)$ denotes the $\boxed{\text{density}}$ of the graph $d(G) = \frac{|E|}{\binom{n}{2}}$.

Definition

We say a graph $G = (V, E)$ is $\boxed{(c, h)\text{-typical}}$ if for every subset $S \subseteq V$ with $|S| \leq h$, we have that

$$\deg_G(S) = (1 \pm |S|c)d(G)^{|S|}n.$$

Quick Review — Definitions

Definition

Recall that $d(G)$ denotes the $\boxed{\text{density}}$ of the graph $d(G) = \frac{|E|}{\binom{n}{2}}$.

Definition

We say a graph $G = (V, E)$ is $\boxed{(c, h)\text{-typical}}$ if for every subset $S \subseteq V$ with $|S| \leq h$, we have that

$$\deg_G(S) = (1 \pm |S|c)d(G)^{|S|}n.$$

Definition

A graph $G = (V, E)$ is $\boxed{\text{tridivisible}}$ if it satisfies the divisibility requirements for a triangle decomposition:

(i) $3 \mid |E|$, and

(ii) $2 \mid \deg(v)$ for all $v \in V$.

Quick Review — Main result

We are proving the following theorem:

Theorem

There exists $0 < c_0 < 1$ and $n_0 \in \mathbb{N}$ such that if $n > n_0$, and G is a $(c, 16)$-typical tridivisible graph on n vertices, with

Quick Review — Main result

We are proving the following theorem:

Theorem

There exists $0 < c_0 < 1$ and $n_0 \in \mathbb{N}$ such that if $n > n_0$, and G is a $(c, 16)$-typical tridivisible graph on n vertices, with

(i) $d(G) > n^{-10^{-7}}$, *and*

Quick Review — Main result

We are proving the following theorem:

Theorem

There exists $0 < c_0 < 1$ *and* $n_0 \in \mathbb{N}$ *such that if* $n > n_0$, *and* G *is a* $(c, 16)$*-typical tridivisible graph on* n *vertices, with*

(i) $d(G) > n^{-10^{-7}}$, *and*

(ii) $c < c_0 d(G)^{10^6}$,

Quick Review — Main result

We are proving the following theorem:

Theorem

There exists $0 < c_0 < 1$ *and* $n_0 \in \mathbb{N}$ *such that if* $n > n_0$, *and* G *is a* $(c, 16)$*-typical tridivisible graph on* n *vertices, with*

(i) $d(G) > n^{-10^{-7}}$, *and*

(ii) $c < c_0 d(G)^{10^6}$,

then G *has a triangle decomposition.*

Quick Review — Strategy

1. Template — Start with SET with some nice structure (G^*).
2. Nibble
3. Cover
4. Hole
5. Completion

Quick Review — Strategy

1. Template — Start with SET with some nice structure (G^*).
2. Nibble — Find large SET in $G \setminus G^*$ using Rödl nibble.
3. Cover
4. Hole
5. Completion

Quick Review — Strategy

1. Template — Start with SET with some nice structure (G^*).
2. Nibble — Find large SET in $G \setminus G^*$ using Rödl nibble.
3. Cover — Get remaining edges into $\triangle$s, but make a mess.
4. Hole
5. Completion

Quick Review — Strategy

1. Template — Start with SET with some nice structure (G^*).
2. Nibble — Find large SET in $G \setminus G^*$ using Rödl nibble.
3. Cover — Get remaining edges into $\triangle$s, but make a mess.
4. Hole — Clean up the mess.
5. Completion

Quick Review — Strategy

1. Template — Start with SET with some nice structure (G^*).
2. Nibble — Find large SET in $G \setminus G^*$ using Rödl nibble.
3. Cover — Get remaining edges into $\triangle$s, but make a mess.
4. Hole — Clean up the mess.
5. Completion — Patch it together to find $\triangle$-decomposition.

Where we stand

Last week, we used the **template** step to get a subgraph $G^* \subseteq G$ such that:

Where we stand

Last week, we used the **template** step to get a subgraph $G^* \subseteq G$ such that:

- $d(G^*) = (1 \pm 3c)d(G)^3\gamma$.

Where we stand

Last week, we used the **template** step to get a subgraph $G^* \subseteq G$ such that:

- $d(G^*) = (1 \pm 3c)d(G)^3\gamma$.
- (G^*, G) is jointly $(c_1, 16)$-typical. (with $c_1 = 6c$)

Where we stand

Last week, we used the **template** step to get a subgraph $G^* \subseteq G$ such that:

- $d(G^*) = (1 \pm 3c)d(G)^3\gamma$.
- (G^*, G) is jointly $(c_1, 16)$-typical. (with $c_1 = 6c$)
- $G \setminus G^*$ is $(c_2, 2)$-typical. (with $c_2 = 50c$)

Goal this week

Definition

Let $J \subseteq G$ with $V(J) = V(G) =: V$, and $E(J) \subseteq E(G)$. We say J is $\boxed{c\text{-bounded}}$ if $\deg_J(v) \leq c|V|$ for all $v \in V$.

Goal this week

Definition

Let $J \subseteq G$ with $V(J) = V(G) =: V$, and $E(J) \subseteq E(G)$. We say J is $\boxed{c\text{-bounded}}$ if $\deg_J(v) \leq c|V|$ for all $v \in V$.

This week, we hope to achieve the following:

Goal this week

Definition

Let $J \subseteq G$ with $V(J) = V(G) =: V$, and $E(J) \subseteq E(G)$. We say J is $\boxed{c\text{-bounded}}$ if $\deg_J(v) \leq c|V|$ for all $v \in V$.

This week, we hope to achieve the following:

- Use the Rödl Nibble to find a SET (set of edge disjoint triangles) $N \subseteq (G \setminus G^*)$ such that $L := (G \setminus G^*) \setminus (\bigcup N)$ is c_3-bounded with $c_3 = c_2^{1/4} = (50c)^{1/4}$.

Goal this week

Definition

Let $J \subseteq G$ with $V(J) = V(G) =: V$, and $E(J) \subseteq E(G)$. We say J is $\boxed{c\text{-bounded}}$ if $\deg_J(v) \leq c|V|$ for all $v \in V$.

This week, we hope to achieve the following:

- Use the Rödl Nibble to find a SET (set of edge disjoint triangles) $N \subseteq (G \setminus G^*)$ such that $L := (G \setminus G^*) \setminus (\bigcup N)$ is c_3-bounded with $c_3 = c_2^{1/4} = (50c)^{1/4}$.
- Find a SET M^C such that $S := \bigcup M^C \cap G^*$ is c_4-bounded with $c_4 = \dfrac{10c_3}{d(G^*)^2}$.

Nibble — What we need

Theorem

There are $c_0 > 0$ *and* $n_0 \in \mathbb{N}$ *such that if* $n > n_0$, $n^{-1/10} < c < c_0$, *and* H *is a* $(c, 2)$*-typical graph on* n *vertices with* $d(H) > \frac{1}{2} n^{-10^{-7}}$, *there is a SET* N *such that* $H \setminus (\bigcup N)$ *is* c'*-bounded for some* $c' < c^{1/4}$.

Nibble — What we need

Theorem

There are $c_0 > 0$ *and* $n_0 \in \mathbb{N}$ *such that if* $n > n_0$, $n^{-1/10} < c < c_0$, *and* H *is a* $(c, 2)$*-typical graph on* n *vertices with* $d(H) > \frac{1}{2} n^{-10^{-7}}$, *there is a SET* N *such that* $H \setminus (\bigcup N)$ *is* c'*-bounded for some* $c' < c^{1/4}$.

Remark

We apply this theorem with $H = (G \setminus G^*)$ and $c = c_2$ to achieve the necessary c_3.

Nibble — What we need

Theorem

There are $c_0 > 0$ *and* $n_0 \in \mathbb{N}$ *such that if* $n > n_0$, $n^{-1/10} < c < c_0$, *and* H *is a* $(c, 2)$*-typical graph on* n *vertices with* $d(H) > \frac{1}{2} n^{-10^{-7}}$, *there is a SET* N *such that* $H \setminus (\bigcup N)$ *is* c'*-bounded for some* $c' < c^{1/4}$.

Remark

We apply this theorem with $H = (G \setminus G^*)$ and $c = c_2$ to achieve the necessary c_3.

Remark

This is a *stronger* conclusion than we achieved with the Rödl Nibble we saw in class: c'-bounded means *every* vertex has degree at most $c'n$ in the leave $H \setminus (\bigcup N)$.

Nibble — Setup

Let $\eta = n^{-10^{-5}}$. Start with $G_0 := H$. Given a graph G_i with $0 \leq i \leq t_0$, we perform the following operation:

Nibble — Setup

Let $\eta = n^{-10^{-5}}$. Start with $G_0 := H$. Given a graph G_i with $0 \leq i \leq t_0$, we perform the following operation:

- Set $d_i := d(H)(1-\eta)^i$.

Nibble — Setup

Let $\eta = n^{-10^{-5}}$. Start with $G_0 := H$. Given a graph G_i with $0 \leq i \leq t_0$, we perform the following operation:

- Set $d_i := d(H)(1-\eta)^i$.
- Set $p_i := \frac{\eta}{d_i^2 n}$.

Nibble — Setup

Let $\eta = n^{-10^{-5}}$. Start with $G_0 := H$. Given a graph G_i with $0 \leq i \leq t_0$, we perform the following operation:

- Set $d_i := d(H)(1-\eta)^i$.
- Set $p_i := \frac{\eta}{d_i^2 n}$.
- Let $\mathcal{T}_i$ be a uniform randomly chosen subset of the triangles of G_i, where each triangle t is in $\mathcal{T}_i$ with probability p_i.

Nibble — Setup

Let $\eta = n^{-10^{-5}}$. Start with $G_0 := H$. Given a graph G_i with $0 \leq i \leq t_0$, we perform the following operation:

- Set $d_i := d(H)(1-\eta)^i$.
- Set $p_i := \frac{\eta}{d_i^2 n}$.
- Let $\mathcal{T}_i$ be a uniform randomly chosen subset of the triangles of G_i, where each triangle t is in $\mathcal{T}_i$ with probability p_i.
- Let $\hat{\mathcal{T}}_i := \mathcal{T}_i \setminus \{\text{all triangles that share edges}\}$.

Nibble — Setup

Let $\eta = n^{-10^{-5}}$. Start with $G_0 := H$. Given a graph G_i with $0 \le i \le t_0$, we perform the following operation:

- Set $d_i := d(H)(1-\eta)^i$.
- Set $p_i := \dfrac{\eta}{d_i^2 n}$.
- Let $\mathcal{T}_i$ be a uniform randomly chosen subset of the triangles of G_i, where each triangle t is in $\mathcal{T}_i$ with probability p_i.
- Let $\hat{\mathcal{T}}_i := \mathcal{T}_i \setminus \{\text{all triangles that share edges}\}$.
- Delete $E\left(\hat{\mathcal{T}}_i\right)$ from G_i to create G_{i+1}.

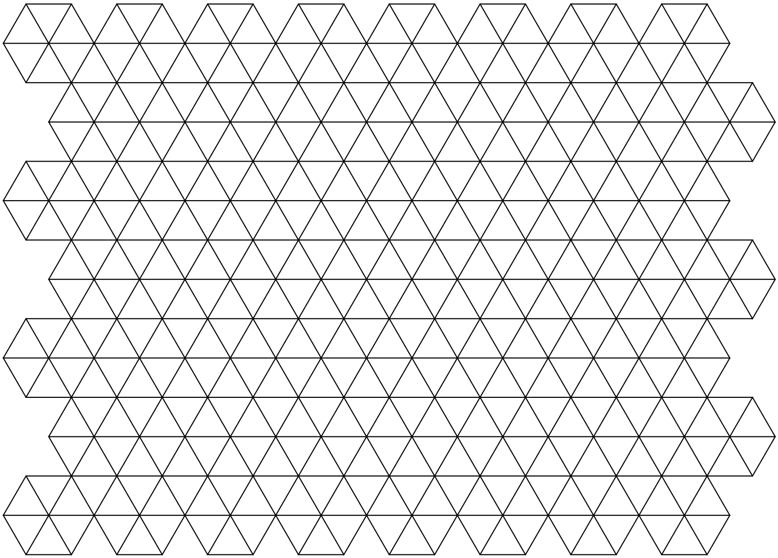
Nibble — Setup

Let $\eta = n^{-10^{-5}}$. Start with $G_0 := H$. Given a graph G_i with $0 \leq i \leq t_0$, we perform the following operation:

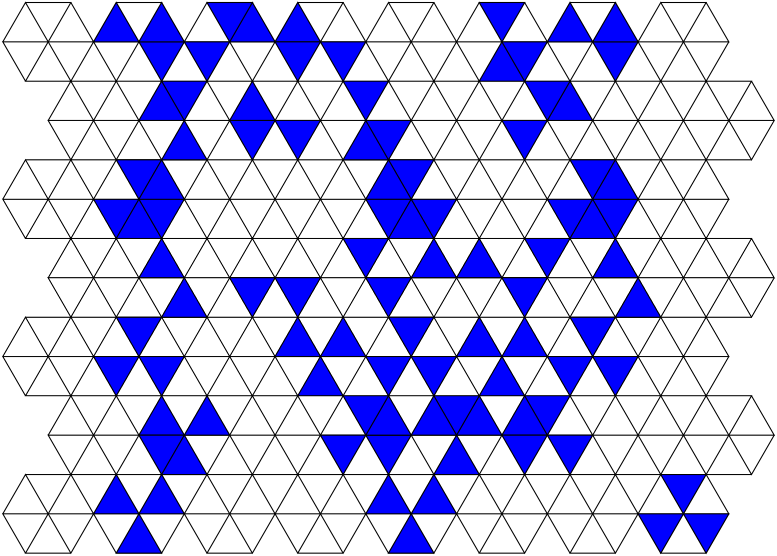
- Set $d_i := d(H)(1-\eta)^i$.
- Set $p_i := \dfrac{\eta}{d_i^2 n}$.
- Let $\mathcal{T}_i$ be a uniform randomly chosen subset of the triangles of G_i, where each triangle t is in $\mathcal{T}_i$ with probability p_i.
- Let $\hat{\mathcal{T}}_i := \mathcal{T}_i \setminus \{\text{all triangles that share edges}\}$.
- Delete $E\left(\hat{\mathcal{T}}_i\right)$ from G_i to create G_{i+1}.

$$\text{Let} \quad N := \bigcup_{i=0}^{t_0} \hat{\mathcal{T}}_i.$$

Simplified example — not to scale

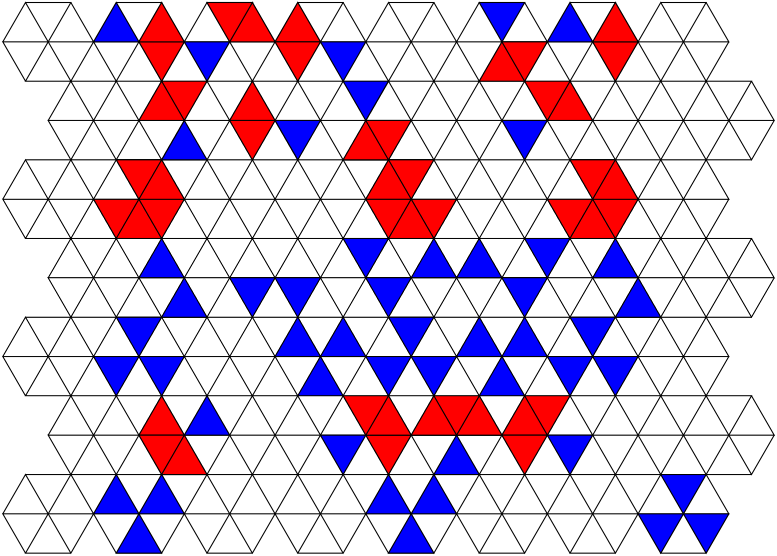


Simplified example — not to scale



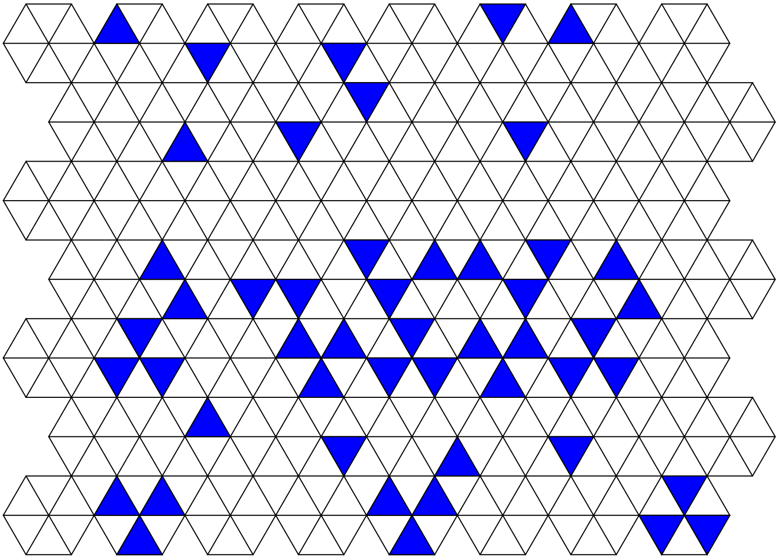
Choose triangles to be in $\mathcal{T}_i$ uniformly at random

Simplified example — not to scale



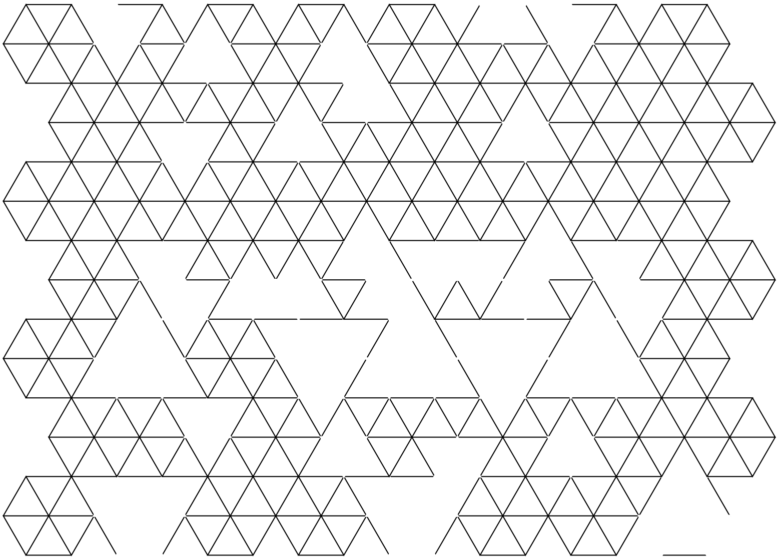
We don't want to keep triangles which share edges

Simplified example — not to scale

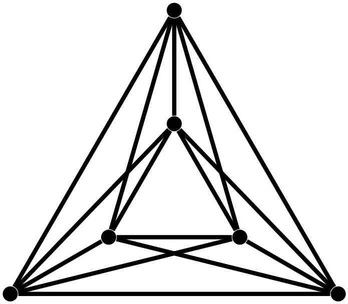


Throw these back to create $\hat{\mathcal{T}}_i$

Simplified example — not to scale



Delete edges in triangles of $\hat{\mathcal{T}}_i$ to create G_{i+1}



Discard all of these triangles.

Nibble Step Lemma

Lemma

Let G_i be a graph such that for all $u, v \in V$,

Nibble Step Lemma

Lemma

Let G_i be a graph such that for all $u, v \in V$,

(i) $\deg_{G_i}(v) = \left(1 \pm c^{(i)}\right) d_i n$, *and*

Nibble Step Lemma

Lemma

Let G_i be a graph such that for all $u, v \in V$,

(i) $\deg_{G_i}(v) = \left(1 \pm c^{(i)}\right) d_i n$, *and*

(ii) $\deg_{G_i}(\{u, v\}) = \left(1 \pm c^{(i)}\right) d_i^2 n$,

Nibble Step Lemma

Lemma

Let G_i be a graph such that for all $u, v \in V$,

(i) $\deg_{G_i}(v) = \left(1 \pm c^{(i)}\right) d_i n$, *and*

(ii) $\deg_{G_i}(\{u, v\}) = \left(1 \pm c^{(i)}\right) d_i^2 n$,

and let G_{i+1} be the graph obtained after the next step of the algorithm. With probability $1 - o\left(\frac{1}{n}\right)$, the following holds for all $u, v \in V$:

Nibble Step Lemma

Lemma

Let G_i be a graph such that for all $u, v \in V$,

(i) $\deg_{G_i}(v) = \left(1 \pm c^{(i)}\right) d_i n$, *and*

(ii) $\deg_{G_i}(\{u, v\}) = \left(1 \pm c^{(i)}\right) d_i^2 n$,

and let G_{i+1} be the graph obtained after the next step of the algorithm. With probability $1 - o\left(\frac{1}{n}\right)$, the following holds for all $u, v \in V$:

(a) $\deg_{G_{i+1}}(v) = \left(1 \pm c^{(i+1)}\right)(1 - \eta) d_i n$,

Nibble Step Lemma

Lemma

Let G_i be a graph such that for all $u, v \in V$,

(i) $\deg_{G_i}(v) = \left(1 \pm c^{(i)}\right) d_i n$, *and*

(ii) $\deg_{G_i}(\{u, v\}) = \left(1 \pm c^{(i)}\right) d_i^2 n$,

and let G_{i+1} be the graph obtained after the next step of the algorithm. With probability $1 - o\left(\frac{1}{n}\right)$, the following holds for all $u, v \in V$:

(a) $\deg_{G_{i+1}}(v) = \left(1 \pm c^{(i+1)}\right)(1 - \eta) d_i n$,

(b) $\deg_{G_{i+1}}(\{u, v\}) = \left(1 \pm c^{(i+1)}\right)((1 - \eta) d_i)^2 n$,

Nibble Step Lemma

Lemma

Let G_i be a graph such that for all $u, v \in V$,

(i) $\deg_{G_i}(v) = \left(1 \pm c^{(i)}\right) d_i n$, *and*

(ii) $\deg_{G_i}(\{u, v\}) = \left(1 \pm c^{(i)}\right) d_i^2 n$,

and let G_{i+1} be the graph obtained after the next step of the algorithm. With probability $1 - o\left(\frac{1}{n}\right)$, the following holds for all $u, v \in V$:

(a) $\deg_{G_{i+1}}(v) = \left(1 \pm c^{(i+1)}\right)(1 - \eta) d_i n$,

(b) $\deg_{G_{i+1}}(\{u, v\}) = \left(1 \pm c^{(i+1)}\right)((1 - \eta) d_i)^2 n$,

(c) *the number of "chosen" and kept triangles is* $\left(1 \pm c^{(i+1)}\right) \eta d_i n^2 / 6$,

Nibble Step Lemma

Lemma

Let G_i be a graph such that for all $u, v \in V$,

(i) $\deg_{G_i}(v) = \left(1 \pm c^{(i)}\right) d_i n$, *and*

(ii) $\deg_{G_i}(\{u, v\}) = \left(1 \pm c^{(i)}\right) d_i^2 n$,

and let G_{i+1} be the graph obtained after the next step of the algorithm. With probability $1 - o\left(\frac{1}{n}\right)$, the following holds for all $u, v \in V$:

(a) $\deg_{G_{i+1}}(v) = \left(1 \pm c^{(i+1)}\right)(1 - \eta) d_i n$,

(b) $\deg_{G_{i+1}}(\{u, v\}) = \left(1 \pm c^{(i+1)}\right)((1 - \eta) d_i)^2 n$,

(c) *the number of "chosen" and kept triangles is* $\left(1 \pm c^{(i+1)}\right) \eta d_i n^2 / 6$,

for $c^{(i+1)} = \left(1 + O\left(c^{(i)} \eta + \eta^2\right)\right) c^{(i)}$.

Nibble Step Proof (very) Sketch

- We bound the probability that $e \in E(G_i)$ is removed.

Nibble Step Proof (very) Sketch

- We bound the probability that $e \in E(G_i)$ is removed.
- For two edges e_1, e_2, we bound the probability that both are removed.

Nibble Step Proof (very) Sketch

- We bound the probability that $e \in E(G_i)$ is removed.
- For two edges e_1, e_2, we bound the probability that both are removed.
- We use these bounds to calculate the expected degree and codegree of vertices in G_{i+1}.

Nibble Step Proof (very) Sketch

- We bound the probability that $e \in E(G_i)$ is removed.
- For two edges e_1, e_2, we bound the probability that both are removed.
- We use these bounds to calculate the expected degree and codegree of vertices in G_{i+1}.
- We need a stronger concentration inequality than Chebychev's to succeed with our stronger conditions.

Nibble Step Proof (very) Sketch

- We bound the probability that $e \in E(G_i)$ is removed.
- For two edges e_1, e_2, we bound the probability that both are removed.
- We use these bounds to calculate the expected degree and codegree of vertices in G_{i+1}.
- We need a stronger concentration inequality than Chebychev's to succeed with our stronger conditions.
- Application of these inequalities gives us the conclusions for degree and codegree, and

Nibble Step Proof (very) Sketch

- We bound the probability that $e \in E(G_i)$ is removed.
- For two edges e_1, e_2, we bound the probability that both are removed.
- We use these bounds to calculate the expected degree and codegree of vertices in G_{i+1}.
- We need a stronger concentration inequality than Chebychev's to succeed with our stronger conditions.
- Application of these inequalities gives us the conclusions for degree and codegree, and
- a degree sum on the resulting graph gives us the number of triangles we removed.

Applying the lemma

- We now apply the lemma iteratively until we reach c_3-boundedness for $(G \setminus G^*) \setminus N$.

Applying the lemma

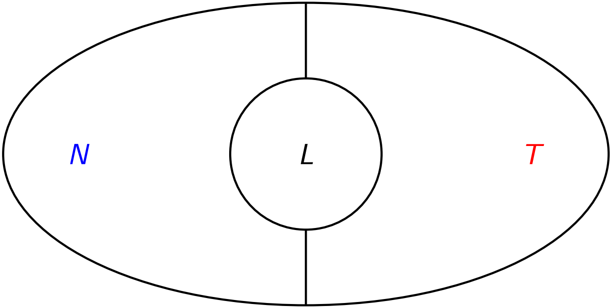
- We now apply the lemma iteratively until we reach c_3-boundedness for $(G \setminus G^*) \setminus N$.
- Note that at step i, the maximum degree in $(G \setminus G^*) \setminus N$ is $d_i\bigl(1 + c^{(i)}\bigr)n \leq 2d_i n$.

Applying the lemma

- We now apply the lemma iteratively until we reach c_3-boundedness for $(G \setminus G^*) \setminus N$.
- Note that at step i, the maximum degree in $(G \setminus G^*) \setminus N$ is $d_i\big(1 + c^{(i)}\big)n \leq 2d_in$.
- Thus, we succeed at attaining our desired boundedness by reducing d_i without letting $c^{(i)}$ run out of control.

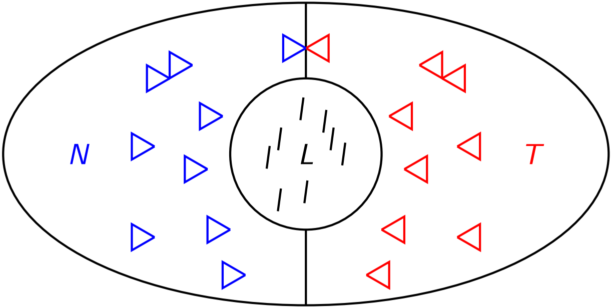
Cover — Idea

Between the **Template** (whose union is G^*) and the **Nibble**, we have all but an extremely small number ($< 3c^{1/4}n^2$) of edges covered by disjoint triangles. The remaining edges lie in L (the **leave**).



Cover — Idea

Between the **Template** (whose union is G^*) and the **Nibble**, we have all but an extremely small number ($< 3c^{1/4}n^2$) of edges covered by disjoint triangles. The remaining edges lie in L (the **leave**).



L is c_3-bounded: max degree in L is at most $c_3 n$

Cover — Idea

Use edges from G^* to cover edges from the **leave**.

Cover — Idea

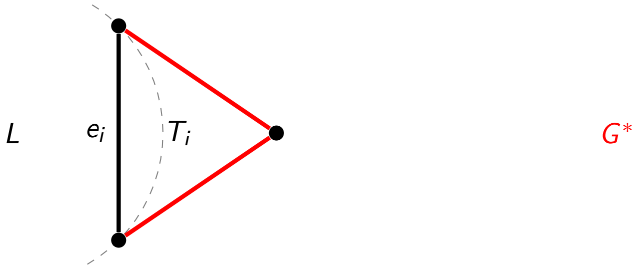
We order the edges: $L = (e_i : i \in [t])$, where $t := |L|$.

Cover — Idea

We want to construct an SET M^C one triangle at at time.

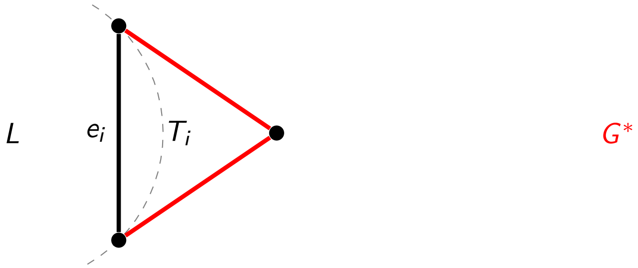
Cover — Idea

For $1 \leq i \leq t$, let T_i be a triangle chosen uniformly at random from those which contain e_i and two edges from G^*…



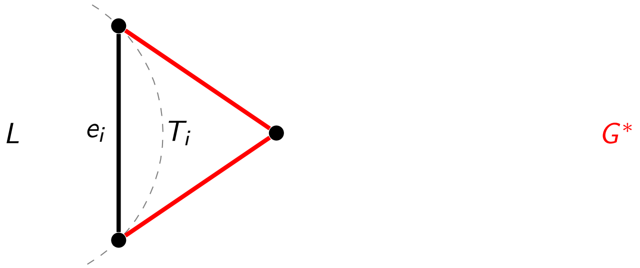
Cover — Idea

... but we *restrict our choices to edges which have not been previously chosen!*



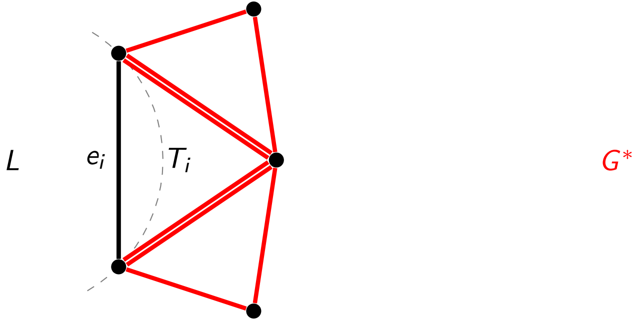
Cover — Idea

If we succeed, M^C will be an SET covering all edges of L.



Cover — Idea

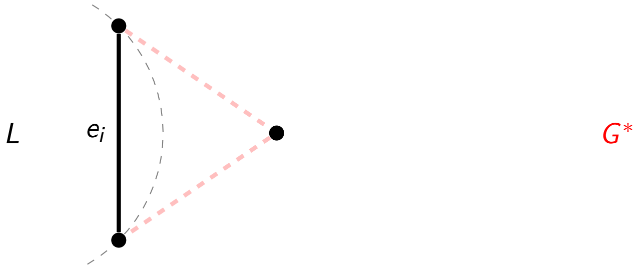
Some edges are now in a triangle of M^C, and another of T, but no edge is in more.



Cover — Idea

Question

What if we want to cover e_i, but there are no appropriate edges of G^* which have not yet been used?



Cover — What could go wrong?

Question

What if we want to cover e_i, but there are no appropriate edges of G^* which have not yet been used?

Cover — What could go wrong?

Question

What if we want to cover e_i, but there are no appropriate edges of G^* which have not yet been used?

Remark

Let $S := \bigcup M^C \cap G^*$. We will require $\deg_S(v)$ to be bounded for all $v \in V$ in order to repair it later.

Cover — What could go wrong?

Question

What if we want to cover e_i, but there are no appropriate edges of G^* which have not yet been used?

Remark

Let $S := \bigcup M^C \cap G^*$. We will require $\deg_S(v)$ to be bounded for all $v \in V$ in order to repair it later.

Remark

We call S the **spill**.

Cover Lemma

Lemma

Following this procedure, for $1 \leq i \leq t$, with high probability, we find an appropriate T_i for each e_i, and also $S = \bigcup M^C \cap G^*$ is c_4-bounded with

$$c_4 = \frac{10c_3}{d(G^*)^2}.$$

Proof of Cover Lemma

Definition

For each $i \in [t]$, let

$$S_i = \bigcup_{j<i} (T_j \setminus e_j).$$

This is the spill from previous triangles at step i.

Proof of Cover Lemma

Definition

For each $i \in [t]$, let

$$S_i = \bigcup_{j<i} (T_j \setminus e_j).$$

This is the spill from previous triangles at step i.

Claim

If S_i is c_4-bounded, we can always find a pair of edges of G^* to create T_i with e_i.

Proof of Cover Lemma

Definition

For each $i \in [t]$, let

$$S_i = \bigcup_{j<i} (T_j \setminus e_j).$$

This is the spill from previous triangles at step i.

Claim

If S_i is c_4-bounded, we can always find a pair of edges of G^* to create T_i with e_i.

Note that if S_i is c_4-bounded, then we have many choices for T_i for each e_i.

Proof of Cover Lemma

- Let B_i be the "bad event" that S_i is not c_4-bounded.

Proof of Cover Lemma

- Let B_i be the "bad event" that S_i is not c_4-bounded.
- Define τ as the minimum i such that B_i holds. Say $\tau = \infty$ if there is no such i.

Proof of Cover Lemma

- Let B_i be the "bad event" that S_i is not c_4-bounded.
- Define τ as the minimum i such that B_i holds. Say $\tau = \infty$ if there is no such i.
- Condition on i:

$$\mathbb{P}(\tau < \infty) \leq \sum_{i=1}^{t} \mathbb{P}(\tau = i \mid \tau \geq i)$$

Proof of Cover Lemma

- Let B_i be the "bad event" that S_i is not c_4-bounded.
- Define τ as the minimum i such that B_i holds. Say $\tau = \infty$ if there is no such i.
- Condition on i:

$$\mathbb{P}(\tau < \infty) \leq \sum_{i=1}^{t} \mathbb{P}(\tau = i \mid \tau \geq i)$$

- Consider a fixed i_0. We want a bound on $\mathbb{P}(\tau = i_0 \mid \tau \geq i_0)$.

Proof of Cover Lemma

- Consider a fixed i_0. We want a bound on $\mathbb{P}(\tau = i_0 \mid \tau \geq i_0)$.

Proof of Cover Lemma

- Consider a fixed i_0. We want a bound on $\mathbb{P}(\tau = i_0 \mid \tau \geq i_0)$.
- We can assume S_{i_0-1} is c_4-bounded (since $\tau \geq i_0$).

Proof of Cover Lemma

- Consider a fixed i_0. We want a bound on $\mathbb{P}(\tau = i_0 \mid \tau \geq i_0)$.
- We can assume S_{i_0-1} is c_4-bounded (since $\tau \geq i_0$).
- In fact, we will show that w.h.p., S_{i_0-1} is $\frac{c_4}{2}$-bounded.

Proof of Cover Lemma

- Consider a fixed i_0. We want a bound on $\mathbb{P}(\tau = i_0 \mid \tau \geq i_0)$.
- We can assume S_{i_0-1} is c_4-bounded (since $\tau \geq i_0$).
- In fact, we will show that w.h.p., S_{i_0-1} is $\frac{c_4}{2}$-bounded.
- Fix some vertex v.

Proof of Cover Lemma

- Consider a fixed i_0. We want a bound on $\mathbb{P}(\tau = i_0 \mid \tau \geq i_0)$.
- We can assume S_{i_0-1} is c_4-bounded (since $\tau \geq i_0$).
- In fact, we will show that w.h.p., S_{i_0-1} is $\frac{c_4}{2}$-bounded.
- Fix some vertex v.
- Define a r.v. X to be the number of edges $e_j \in L$ (with $j < i_0$) such that T_j uses e_j, and two e_j-v edges.

Proof of Cover Lemma

- Consider a fixed i_0. We want a bound on $\mathbb{P}(\tau = i_0 \mid \tau \geq i_0)$.
- We can assume S_{i_0-1} is c_4-bounded (since $\tau \geq i_0$).
- In fact, we will show that w.h.p., S_{i_0-1} is $\frac{c_4}{2}$-bounded.
- Fix some vertex v.
- Define a r.v. X to be the number of edges $e_j \in L$ (with $j < i_0$) such that T_j uses e_j, and two e_j-v edges.
- $\deg_{S_{i_0-1}}(v) \leq \deg_L(v) + 2X$.

Proof of Cover Lemma

- Consider a fixed i_0. We want a bound on $\mathbb{P}(\tau = i_0 \mid \tau \geq i_0)$.
- We can assume S_{i_0-1} is c_4-bounded (since $\tau \geq i_0$).
- In fact, we will show that w.h.p., S_{i_0-1} is $\frac{c_4}{2}$-bounded.
- Fix some vertex v.
- Define a r.v. X to be the number of edges $e_j \in L$ (with $j < i_0$) such that T_j uses e_j, and two e_j-v edges.
- $\deg_{S_{i_0-1}}(v) \leq \deg_L(v) + 2X$.
- We want to bound X.

Proof of Cover Lemma

- Consider a fixed i_0. We want a bound on $\mathbb{P}(\tau = i_0 \mid \tau \geq i_0)$.
- We can assume S_{i_0-1} is c_4-bounded (since $\tau \geq i_0$).
- In fact, we will show that w.h.p., S_{i_0-1} is $\frac{c_4}{2}$-bounded.
- Fix some vertex v.
- Define a r.v. X to be the number of edges $e_j \in L$ (with $j < i_0$) such that T_j uses e_j, and two e_j-v edges.
- $\deg_{S_{i_0-1}}(v) \leq \deg_L(v) + 2X$.
- We want to bound X.
- $X = \sum_{j<i_0} X_j$, where X_j is the indicator r.v. for the event $T_j = e_j \cup \{v\}$.

Proof of Cover Lemma

Definition

A random variable $Y = \sum_{i=1}^{n} Y_i$ is (μ, C)-dominated if:

Proof of Cover Lemma

Definition

A random variable $Y = \sum_{i=1}^{n} Y_i$ is (μ, C)-dominated if:

(i) $|Y_i| \leq C$ for every $1 \leq i \leq n$, and

Proof of Cover Lemma

Definition

A random variable $Y = \sum_{i=1}^{n} Y_i$ is (μ, C)-dominated if:

(i) $|Y_i| \leq C$ for every $1 \leq i \leq n$, and

(ii) there are $\mu_1, \ldots, \mu_n$ such that $\sum \mu_i \leq \mu$, and conditional on any values Y_j for $j < i$, we have $\mathbb{E}[|Y_i|] \leq \mu_i$ for every $1 \leq i \leq n$.

Proof of Cover Lemma

Definition

A random variable $Y = \sum_{i=1}^{n} Y_i$ is (μ, C)-dominated if:

(i) $|Y_i| \leq C$ for every $1 \leq i \leq n$, and

(ii) there are $\mu_1, \ldots, \mu_n$ such that $\sum \mu_i \leq \mu$, and conditional on any values Y_j for $j < i$, we have $\mathbb{E}[|Y_i|] \leq \mu_i$ for every $1 \leq i \leq n$.

- Number of choices from e_j's perspective is at least $\deg_{G^*}(e_j) - 2c_4 n \geq d(G^*)^2 n/2$ (c_4 is tiny compared to $d(G^*)$).

Proof of Cover Lemma

Definition

A random variable $Y = \sum_{i=1}^{n} Y_i$ is (μ, C)-dominated if:

(i) $|Y_i| \leq C$ for every $1 \leq i \leq n$, and

(ii) there are $\mu_1, \ldots, \mu_n$ such that $\sum \mu_i \leq \mu$, and conditional on any values Y_j for $j < i$, we have $\mathbb{E}[|Y_i|] \leq \mu_i$ for every $1 \leq i \leq n$.

- Number of choices from e_j's perspective is at least $\deg_{G^*}(e_j) - 2c_4 n \geq d(G^*)^2 n/2$ (c_4 is tiny compared to $d(G^*)$).
- $\mathbb{P}(X_j = 1) \leq \dfrac{2}{d(G^*)^2 n} \implies X$ is $\left(\frac{2|L|}{d(G^*)^2 n}, 1\right)$-dominated.

Proof of Cover Lemma

Lemma (Black Box — Keevash)

If Y is $(\mu, 1)$-dominated, then

$$\mathbb{P}(|Y| > 2\mu) \leq 2\exp(-\mu/6)$$

Proof of Cover Lemma

Lemma (Black Box — Keevash)

If Y is $(\mu, 1)$-dominated, then

$$\mathbb{P}(|Y| > 2\mu) \leq 2\exp(-\mu/6)$$

Now, since L is c_3-bounded, we have

$$\mathbb{E}[X] \leq \frac{2|L|}{d(G^*)^2 n} \leq \frac{c_3 n}{d(G^*)^2},$$

Proof of Cover Lemma

Lemma (Black Box — Keevash)

If Y is $(\mu, 1)$-dominated, then

$$\mathbb{P}(|Y| > 2\mu) \leq 2\exp(-\mu/6)$$

Now, since L is c_3-bounded, we have

$$\mathbb{E}[X] \leq \frac{2|L|}{d(G^*)^2 n} \leq \frac{c_3 n}{d(G^*)^2},$$

and we deduce

$$\mathbb{P}\left(X > \frac{2c_3 n}{d(G^*)^2}\right) < 2\exp\left(-\frac{c_3 n}{12 d(G^*)^2}\right) < \exp\left(-\Omega\left(n^{1/2}\right)\right)$$

Proof of Cover Lemma

- Now, we have w.h.p. that $\deg_{S_j}(v) \leq \frac{c_4 n}{2}$ for $j < i_0$.

Proof of Cover Lemma

- Now, we have w.h.p. that $\deg_{S_j}(v) \leq \frac{c_4 n}{2}$ for $j < i_0$.
- There are only polynomially many choices for v.

Proof of Cover Lemma

- Now, we have w.h.p. that $\deg_{S_j}(v) \leq \frac{c_4 n}{2}$ for $j < i_0$.
- There are only polynomially many choices for v.
- So a union bound (over $v \in V$) gives that w.h.p., S_{i_0-1} is $\frac{c_4}{2}$-bounded, and so $\tau \neq i_0$.

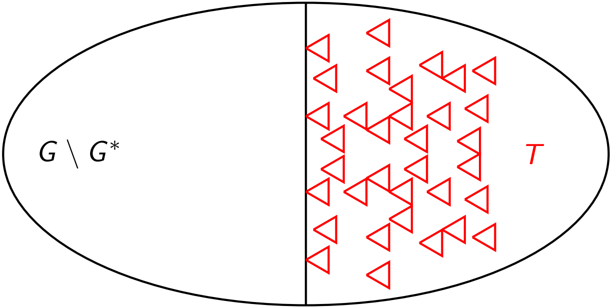
Proof of Cover Lemma

- Now, we have w.h.p. that $\deg_{S_j}(v) \leq \frac{c_4 n}{2}$ for $j < i_0$.
- There are only polynomially many choices for v.
- So a union bound (over $v \in V$) gives that w.h.p., $S_{i_0 - 1}$ is $\frac{c_4}{2}$-bounded, and so $\tau \neq i_0$.
- Now, a union bound over all $1 \leq i_0 \leq t$ gives that w.h.p., $\tau = \infty$, as desired.

Proof of Cover Lemma

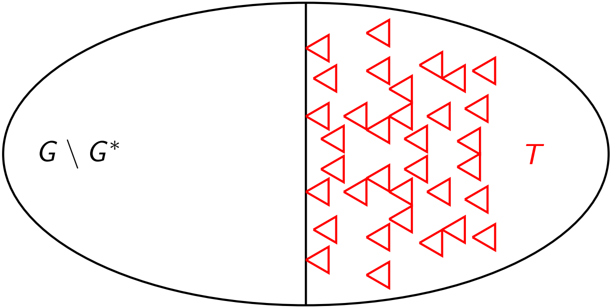
- Now, we have w.h.p. that $\deg_{S_j}(v) \leq \frac{c_4 n}{2}$ for $j < i_0$.
- There are only polynomially many choices for v.
- So a union bound (over $v \in V$) gives that w.h.p., S_{i_0-1} is $\frac{c_4}{2}$-bounded, and so $\tau \neq i_0$.
- Now, a union bound over all $1 \leq i_0 \leq t$ gives that w.h.p., $\tau = \infty$, as desired.
- Thus, we successfully complete the cover of the edges of L, and S is c_4-bounded.

Where we now stand



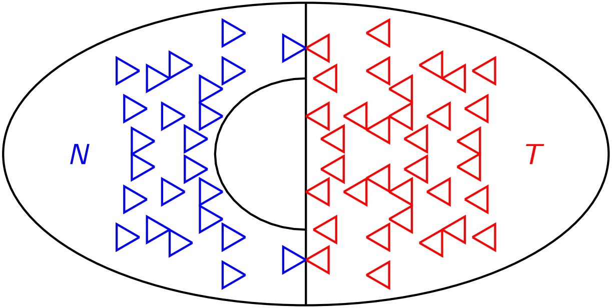
We had completed the **template** step, giving an SET T.

Where we now stand



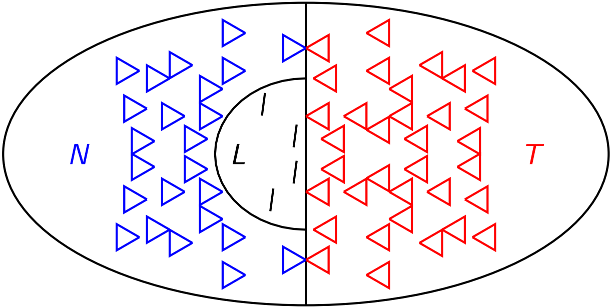
The graph on the uncovered edges is $(c_2, 2)$-typical.

Where we now stand



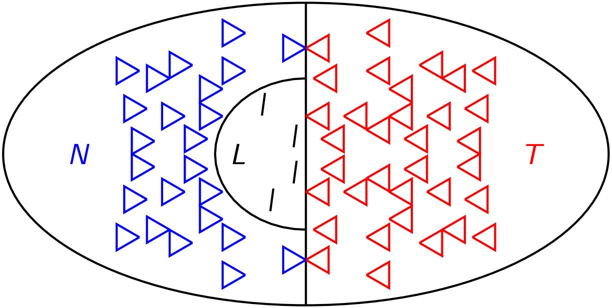
The **nibble** covers most of the remaining edges, but. . .

Where we now stand



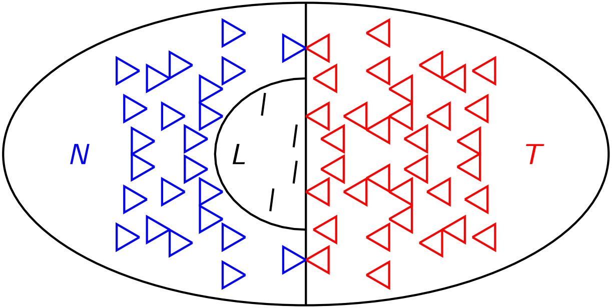
...leaves behind a set L of uncovered edges.

Where we now stand



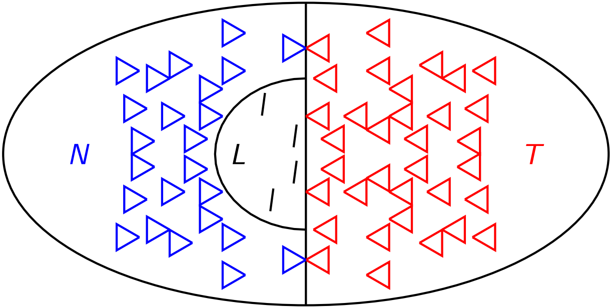
L is c_3-bounded, i.e., $\deg_L(v) \leq c_3 n$ for $v \in V$.

Where we now stand



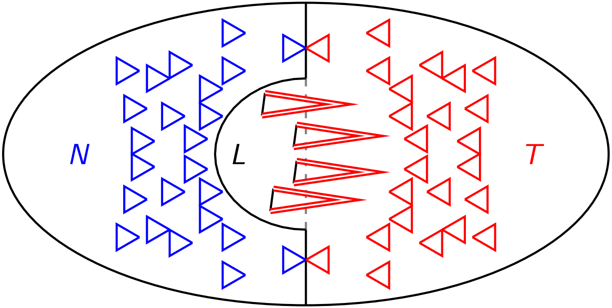
N, L, and T are pairwise disjoint as edge sets, and...

Where we now stand



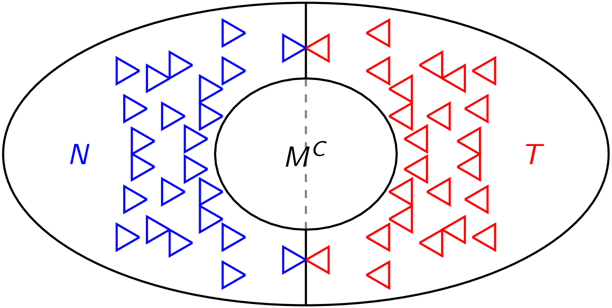
T, N are each an SET.

Where we now stand



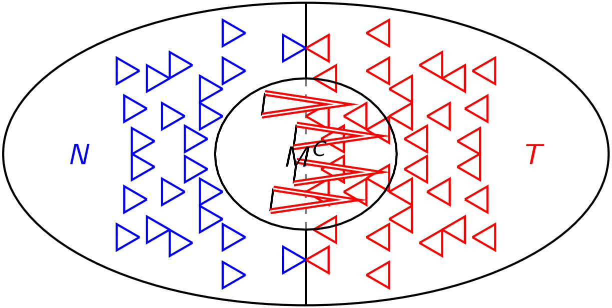
We then **cover** L with triangles using edges from $G^* = \bigcup T$.

Where we now stand



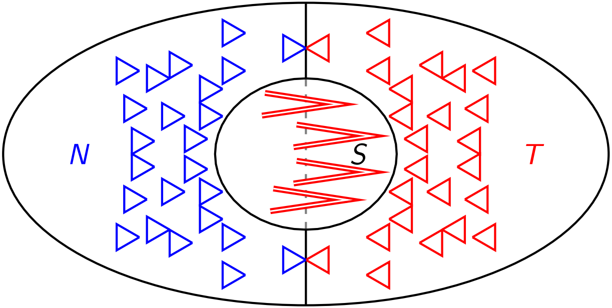
This gives another SET M^C, but...

Where we now stand



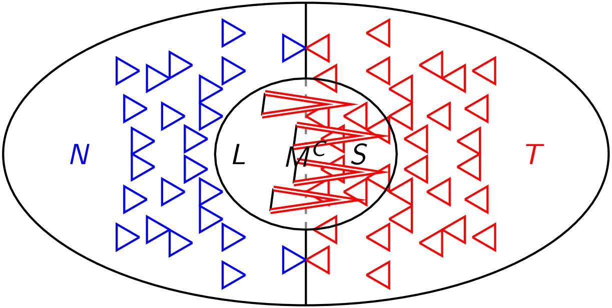
$\dots \left(\bigcup M^C\right) \cap G^* \neq \emptyset.$

Where we now stand



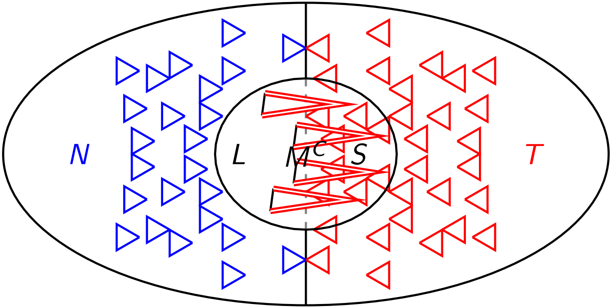
The graph on the set of edges covered twice is denoted S.

Where we now stand



A mess, but S is c_4-bounded: $\deg_S(v) \leq c_4 n$ for all $v \in V$.

Where we now stand



This will allow us to clean up this mess next week.

Where we now stand — All on one page

Recall that the **template** step gave us an SET T with $\bigcup T = G^*$ such that

- $d(G^*) = (1 \pm 3c)d(G)^3\gamma$.
- (G^*, G) is jointly $(c_1, 16)$-typical. (with $c_1 = 6c$)
- $G \setminus G^*$ is $(c_2, 2)$-typical. (with $c_2 = 50c$)

Now, we have covered all edges of G with triangles. We have SETs N, T, M^C with $N \cap M^C = \emptyset$, and $N \cap T = \emptyset$. Sadly, there is a subgraph $S = \left(\bigcup M^C\right) \cap G^*$ of edges which are covered twice, with the following properties:

- S is c_4-bounded.
- S is tridivisible.