Recall the following definition of a k-colored sum-free set:

Definition. Let G be an abelian group and let $k \geq 3$. A *k-colored sum-free set* in G is a collection of k-tuples $(x_{1,i}, \dots, x_{k,i})$, $1 \leq i \leq L$, of elements of G such that, for all $i_1, \dots, i_k \in [L]$, we have

$$x_{1,i_1} + \ldots + x_{k,i_k} = 0 \text{ if and only if } i_1 = i_2 = \ldots = i_k.$$

The size of a k-colored sum-free set is the number L of k-tuples it contains.

The goal of this exercise is to show the following theorem:

Theorem. *Fix $k \geq 3$ and let $m = p^\ell$ for some prime p and $\ell \geq 1$. Then, for any $d \geq 1$, the size of any k-colored sum-free set in $\mathbb{Z}_m^d$ is at most $(\Gamma_{m,k})^d$, where*

$$\Gamma_{m,k} = \min_{0<t<1} \frac{1 + t + \cdots + t^{m-1}}{t^{(m-1)/k}}.$$

We can split the proof into the following steps.

(a) Show that over $\mathbb{F}_p$ we have, for all $z_1, \dots, z_k \in \mathbb{Z}_m$,

$$\sum_{\substack{a_1,\dots,a_k \in \{0,\dots,m-1\} \\ a_1+\cdots+a_k \leq m-1}} \prod_{j=1}^{k} \left((-1)^{a_j} \binom{z_j}{a_j} \right) = \begin{cases} 1 & \text{if } z_1 + \cdots + z_k = 0 \text{ in } \mathbb{Z}_m, \\ 0 & \text{otherwise.} \end{cases}$$

You may use the fact that if $0 \leq a \leq m-1$ and $z_1 \equiv z_2 \pmod{m}$, then $\binom{z_1}{a} \equiv \binom{z_2}{a} \pmod{p}$.

(b) Let $(\delta_1, \dots, \delta_d)$ be uniformly distributed over the set $\{0, 1, \dots, m-1\}^d$. By considering the random variable $t^{\sum_i \delta_i}$, for some appropriate choice of $t \in (0,1)$, show that

$$\left| \left\{ (\delta_1, \dots, \delta_d) \in \{0, 1, \dots, m-1\}^d : \sum_{i=1}^{d} \delta_i \leq \frac{d(m-1)}{k} \right\} \right| \leq (\Gamma_{m,k})^d.$$

(c) Given a k-colored sum-free set in $\mathbb{Z}_m^d$ of size L, use the result from part (a) to construct a diagonal k-tensor $M : [L]^k \to \mathbb{F}_p$. Using the result from part (b), bound its slice rank to show

$$L \leq k \left(\Gamma_{m,k}\right)^d.$$

[For more information on tensors and the slice rank method, see the course notes from Extremal Combinatorics: `http://discretemath.imp.fu-berlin.de/DMII-2019-20/Week11.pdf`.]

(d) Show that, given a k-colored sum-free set in $\mathbb{Z}_m^d$ of size L and a positive integer ℓ, we can construct a k-colored sum-free set of size L^ℓ in $\mathbb{Z}_m^{d\ell}$.

Deduce that $L \leq (\Gamma_{m,k})^d$.

If you are really stuck, you can find some hints at
`http://discretemath.imp.fu-berlin.de/DMIIISem-2019-20/exhints.html`.