Schnyder's Theorem

Presentation

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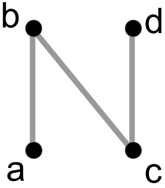
Introduction

- Definitions
  - Linear Extension of a Poset
  - Realizer of a Poset
  - Dimension of a Poset
  - Incidence Poset of a Graph
- Schnyder's Theorem: G is planar $\iff \dim(P_G) \leq 3$
  - $\dim(P_G) \leq 3 \Rightarrow G$ is planar
  - G is planar $\Rightarrow \dim(P_G) \leq 3$

Linear Extension

Definition: A **linear extension** of a poset P is a permutation of its elements $p_1,...,p_n$, such that $p_i < p_j$ implies $i < j$.

Example: $P = \{(a, b), (c, b), (c, d), (a, a), (b, b), (c, c), (d, d)\}$



$L_1 : a < c < b < d$

$L_2 : a < c < d < b$

$L_3 : c < a < b < d$

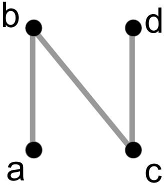
$L_4 : c < a < d < b$

$L_5 : c < d < a < b$

Realizer

Definition: A set $S = \{L_1, L_2, ..., L_t\}$ of linear extensions of a poset P is a **realizer** of P if $\bigcap S = P$. That means whenever x is incomparable to y in P, there is some L_i in F with $x > y$ in L_i.

Example: P $= \{(a,b),(c,b),(c,d),(a,a),(b,b),(c,c),(d,d)\}$



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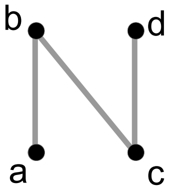
$L_5 : c < d < a < b$

$L_2 \cap L_4 \cap L_5 = P$?

$L_1 \cap L_5 = P$?

Dimension

Definition: The **dimension** of a poset P is the cardinality of the realizer with the fewest elements.
Example: P $= \{(a,b),(c,b),(c,d),(a,a),(b,b),(c,c),(d,d)\}$



$$L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5 = P$$

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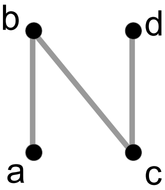
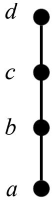


$L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5 = P$

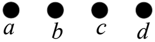
$L_2 \cap L_3 \cap L_5 = P$

$L_1 \cap L_5 = P \quad \Rightarrow dim(P) = 2$

Examples



$$L: a<b<c<d \qquad \Rightarrow dim(P_l)=1$$

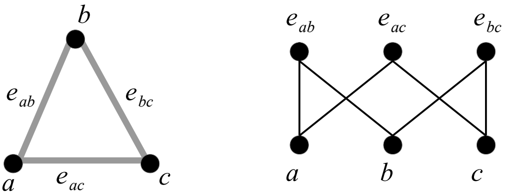


$$L_1: a<b<c<d$$
$$L_1: a<b<d<c$$
$$\vdots$$
$$L_{24}: d<c<b<a$$

$$L_1 \cap L_{24} = P \qquad \Rightarrow dim(P_a)=2$$

Incidence Poset

Definition: If $G = (V, E)$ is a graph, then the **incidence poset** of G is the graded poset where every vertex has rank 1, every edge has rank 2, and for $v \in V$ and $e \in E$ we have $v \leq e$ if e and v are incident.
Example for triangle graph:



Schnyder's Theorem

Let $G = (V, E)$ be a graph and let $P_G = (V \cup E, P)$ be the incidence poset associated with G. Then G is planar if and only if $dim(P_G) \leq 3$.
Proof of G is planar $\iff \dim(P_G) \leq 3$

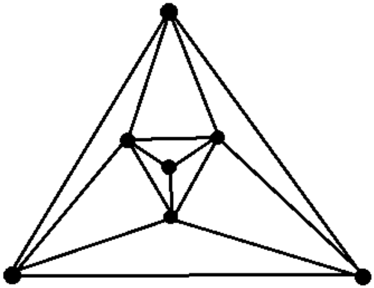
1. $\dim(P_G) \leq 3 \Rightarrow G$ is planar
2. G is planar $\Rightarrow \dim(P_G)$

Proof: $\dim(P_G) \leq 3 \Rightarrow G$ is planar

If G is planar we can draw it in the plane without edge crossings involving edges that do not share an end point.

G is planar $\Rightarrow \dim(P_G) \leq 3$

Assumption: Without loss of generality let G be a maximal planar graph. We use straight line segments for the edges so that our diagram will be a triangulation.



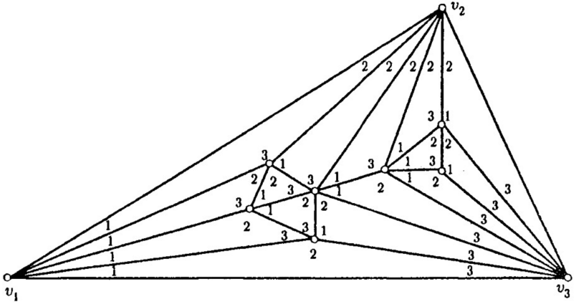
G is planar $\Rightarrow \dim(P_G) \leq 3$

Let f be function that colors all interior angles of G into $1, 2, 3$. We call the function normal coloring or Schnyder labeling if the following properties are satisfied:

1. Angles incident with exterior vertex v_i are colored i, for $i = 1, 2, 3$
2. At each interior vertex u, there will be an angle colored i for $i = 1, 2, 3$
3. At each interior vertex u, all angles with the same color are consecutive
4. At each interior vertex u, the block of angles colored $i + 1$ appears immediatly after the block that is colored i in clockwise order
5. For each elementary triangle the three angles are colored 1,2 and 3 in clockwise order.

G is planar $\Rightarrow \dim(P_G) \leq 3$

Example of a normal colored triangulation:



G is planar $\Rightarrow \dim(P_G) \leq 3$

Claims:

1. Every planar triangulation has a normal coloring.
2. If C is a cycle in T, then C contains a Type i vertex for each $i = 1, 2, 3$.
3. Let P_i be the binary relation on the set V of vertices of G defined by $(x, y) \in P_i$ if and only if the two vertices are end point of the same edge and both angles incident to this edge and to y are colored i, or $x = y$. Then the transitive closure $Q_i = tr(P_i)$ is a partial order.

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Proof of Schnyder's Theorem

1. $\dim(P_G) \leq 3 \Rightarrow G$ is planar $\square$
2. G is planar $\Rightarrow \dim(P_G)$ $\square$

$\Longrightarrow G$ is planar $\Longleftrightarrow \dim(P_G) \leq 3$

Graham; Groetschel; Lovasz(1995): *Handbook of Combinatorics. Volume* 1. North-Holland, Amsterdam, Cambridge, Massachusetts: Elsevier; MIT Press.

Trotter, William T. (1992): *Combinatorics and Partially Ordered Sets. Dimension Theory.* Baltimore: Johns Hopkins Univ. Pr..