Problem Set # 1

Discrete Mathematics III (Probabilistic Method)
WS 2010/11

Tibor Szabó

The first five problems are due on October 26, 14:15pm.

The last five are due on November 2, 14:15pm.

You are welcome to submit **at most** two neatly written exercises each week. You must achieve a total of 15 full solutions during the semester.

You should try to solve all the exercises, because they are part of the final exam.

1. Let $\{(A_i, B_i), 1 \le i \le h\}$ be a family of pairs of subsets of the set of integers such that $|A_i| = k$ for all i and $|B_i| = l$ for all i, $A_i \cap B_i = \emptyset$ and $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$ for all $i \neq j$. Prove that $h \le \frac{(k+l)^{k+l}}{k^k l^l}$.

2. Prove that if there is a real p, $0 \le p \le 1$ such that
$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1,$$
then the Ramsey number $R(k,t)$ satisfies $R(k,t) > n$. Using this, show that
$$R(4,t) \ge \Omega\left(\frac{t^{3/2}}{\ln^{3/2} t}\right).$$

3. The k-color Ramsey number $R_k(H)$ of a graph H is the smallest integer N such that in any k-coloring of the edges of K_N there is a monochromatic copy of H. For example $R_2(K_3) = 6$. Show that for some constants C_1, C_2, we have
$$C_1 \frac{k^3}{\log^3 k} \le R_k(K_{3,3}) \le C_2 k^3.$$

   (*Hint*: You might want to use the fact that the Turán number of $K_{3,3}$ is $\Theta(n^{5/3})$. (The Turán number $ex(n,H)$ of a graph H is the largest integer m, such that there exists an H-free graph on n vertices with m edges.))

4. Let $G = (V, E)$ be a bipartite graph on n vertices with a list $S(v)$ of at least $\log_2 n$ colors associated with each vertex $v \in V$. Prove that there is a proper coloring of G assigning to each vertex v a color from its list $S(v)$.

5. Suppose $n \geq 4$ and let H be an n-uniform hypergraph with at most $\frac{4^{n-1}}{3^n}$ edges. Prove that there is a coloring of the vertices of H by 4 colors so that in every edge all 4 colors are represented.

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6. Prove that there is a positive constant c so that every set A of n nonzero reals contains a subset $B \subset A$ of size $|B| \geq cn$ so that there are no $b_1, b_2, b_3, b_4 \in B$ satisfying

$$b_1 + 2b_2 = 2b_3 + 2b_4.$$

7. $(*)$ Let $G = (V, E)$ be a graph with n vertices and minimum degree $\delta > 10$. Prove that there is a partition of V into two disjoint subsets A and B such that $|A| \leq 4\frac{n \ln \delta}{\delta}$ and every vertex in B has a neighbor in A and has a neighbor in B.

8. Let F be a finite collection of binary strings of finite lengths and assume no member of F is a prefix of another one. Let N_i denote the number of strings of length i in F. Prove that

$$\sum_i \frac{N_i}{2^i} \leq 1.$$

9. Prove that there is an absolute constant $c > 0$ with the following property. Let A be an n by n matrix with pairwise distinct entries. Then there is a permutation of the rows of A so that no column in the permuted matrix contains an increasing sub-sequence of length at least $c\sqrt{n}$.

10. $(*)$ We proved in class that for every integer $k > 0$ there is tournament $T_k = (V, E)$ with $|V| > k$ such that for every set U of at most k vertices of T_k there is a vertex v which dominates U. Show that each such tournament contains at least $\Omega(k2^k)$ vertices.