Problem Set # 11

Discrete Mathematics III (Probabilistic Method)
WS 2010/11

Tibor Szabó

The solutions are due on January 25th, 14:15pm.

You are welcome to submit **at most** two neatly written exercises each week. You must achieve a total of 15 full solutions during the semester.

You should try to solve all the exercises, they are part of the final exam.

1. Let $G = (V, E)$ be a simple graph and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size at least $10d$, where $d \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most d neighbors u of v such that c lies in $S(u)$. Prove that there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$.

2. A Boolean function f is called a (k, s)-CNF formula if f is the AND of an arbitrary (finite) number of clauses, each clause is an OR of exactly k literals (where each literal is either a variable or its negation) of k distinct variables, and each variable appears in at most s clauses. Let $f(k)$ be the largest integer s such that every (k, s)-CNF formula is satisfiable.

   (i) Prove that $f(3) \geq 3$

   (ii) Prove that $f(k) \geq \left\lfloor \frac{2^k}{ke} \right\rfloor - 1$ for all k.

3. (*) Prove that for every integer $d > 1$ there is a finite $c(d)$ such that any bipartite graph with maximum degree d and girth at least $c(d)$ can be properly edge-colored by $d + 1$ colors so that there is no two-colored cycle.