Problem Set # 12

Discrete Mathematics III (Probabilistic Method)
WS 2010/11

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The solutions are due on Februar 1st, 14:15pm.

You are welcome to submit **at most** two neatly written exercises each week. You must achieve a total of 15 full solutions during the semester.

You should try to solve all the exercises, they are part of the final exam.

1. Let G be a d-regular graph on $d^{d^{d^d}}$ vertices. Prove that there exists a spanning subgraph H of G with all degrees roughly half the original: for all $v \in V(H)$,

$$\left| d_H(v) - \frac{d}{2} \right| \leq 10\sqrt{d \log d}.$$

2. Recall the deifinition of the function $f(k)$ from the previous exercise sheet: it is the largest integer s such that every (k,s)-CNF formula is satisfiable. Improve the lower bound by a factor 2 using a non-uniform random assignment: Show that,

$$f(k) \geq \left\lfloor \frac{2^{k+1}}{e(k+1)} \right\rfloor.$$

Hint: In a (k,s)-CNF formula set a variable v true with probability $P_v := \frac{1}{2} + \frac{2d_{\bar{v}}-s}{2sk}$, where d_ℓ represents the number of occurrences of a literal ℓ. Then the negated variable $\bar{v}$ is satisfied with probability $P_{\bar{v}} = \frac{1}{2} - \frac{2d_{\bar{v}}-s}{2sk} \geq \frac{1}{2} + \frac{2d_v - s}{2sk}$, as we have $d_v + d_{\bar{v}} \leq s$. Note the unintuitive nature of this choice: the more a variable v occurs negated the less likely we will satisfy $\bar{v}$.

3. (*) Prove that for every $\epsilon > 0$ there is a finite $l_0 = l_0(\epsilon)$ and an infinite sequence of bits $a_1, a_2, a_3, \ldots \quad a_i \in \{0,1\}$, such that for every $l > l_0$ and every $i \geq 1$ the two binary vectors $u = (a_i, a_{i+1}, \ldots, a_{i+l-1})$ and $v = (a_{i+l}, a_{i+l+1}, \ldots, a_{i+2l-1})$ differ in at least $(\frac{1}{2} - \epsilon)l$ coordinates.