Problem Set # 3

Discrete Mathematics III (Probabilistic Method)
WS 2010/11

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The solutions are due on November 9, 14:15pm.

You are welcome to submit **at most** two neatly written exercises each week. You must achieve a total of 15 full solutions during the semester.

You should try to solve all the exercises, because they are part of the final exam.

1. Let $m(k)$ be the minimum number of edges of a non-two-colorable k-uniform hypergraph, as introduced in the lecture. There we have seen that $2^{k-1} \le m(k) = O(k^2 2^k)$. Show the improved lower bound $\Omega(k^{\frac{1}{4}} 2^k)$ by analysing the following extremely simple coloring procedure. Given a k-uniform hypergraph $\mathcal{F} \subseteq \binom{X}{k}$,

   1. Color each vertex with `blue`
   2. Select a random permutation σ of the vertex set u.a.r.
   3. For each vertex $v \in X$ switch the color of v from `blue` to `red` if v is the first (according to σ) among the vertices of some hyperedge in $\mathcal{F}$.

   Remark To appreciate the above algorithm (due to Pluhár) the history of the problem must be noted. The conjecture that $m(k)/2^k \to \infty$ was suggested by Erdős and Lovász in 1973 in a historic paper (where, among others, they introduce and use an innocent looking probabilistic tool that later goes on to have a fabulous carreer under the name of (Lovász) Local Lemma. We will also see quite a bit of it later in the course.). Their conjecture was confirmed by Beck (1977), who proved a logarithmic lower bound. Later Beck (1978) gave the first polynomial lower bound of $k^{\frac{1}{3}-o(1)}$ using a random recoloring algorithm with a quite intricate analysis (which I would certainly not dare to give as a homework exercise). In the lecture we will soon see an improved version of this algorithm (due to Radhakrishnan and Srinivasan, 2000) giving $\Omega(k^{\frac{1}{2}-o(1)})$. The determination of the correct power of k in $m(k)/2^k$ is an outstanding open question of the field.

2. Let $\mathcal{F}$ be a family of subsets of $[n] = \{1, 2, \ldots, n\}$, and suppose there are no $A, B \in \mathcal{F}$ satisfying $A \subset B$. Let $\sigma \in S_n$ be a random permutation of the elements of $[n]$ and consider the random

variable

$$X = |\{i : \{\sigma(1), \sigma(2), \ldots \sigma(i)\} \in \mathcal{F}\ \}|.$$

By considering the expectation of X prove that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

3. (*) Prove that for every two independent, identically distributed real random variables X and Y,

$$Prob(|X - Y| \leq 2) \leq 3Prob(|X - Y| \leq 1).$$