Problem Set # 5

Discrete Mathematics III (Probabilistic Method)
WS 2010/11

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The solutions are due on November 23, 14:15pm.

You are welcome to submit **at most** two neatly written exercises each week. You must achieve a total of 15 full solutions during the semester.

You should try to solve all the exercises, they are part of the final exam.

1. Prove that the Ramsey number $R(k,k)$ satisfies, for every integer n,

$$R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}},$$

and conclude that

$$R(k,k) \geq (1 - o(1)) \frac{k}{e} 2^{k/2}.$$

(A constant factor $\sqrt{2}$ improvement over the bound of the first lecture.)

2. Let $\mathcal{G}_r(n)$ denote the set of all r-partite graphs with parts $V_1, \ldots, V_r$, where $|V_i| = n$ for all $i = 1, \ldots, r$. Let H be a graph with l vertices, $l \leq r$. A subgraph T of $G \in \mathcal{G}_r(n)$ is called an H-*transversal* if T is isomorphic to H and $|V(T) \cap V_i| \leq 1$ for all $i = 1, \ldots, r$. The *r-partite Turán number* $ex_r(n,H)$ is the largest integer m such that there exists a graph $G \in \mathcal{G}_r(n)$ with m edges containing no H-free transversal. (For example, $ex_n(1,H)$ is just the ordinary Turán number $ex(n,H)$.)

Prove that

$$ex_r(n,H) = ex(r,H)n^2.$$

3. (*) Let X be a collection of pairwise orthogonal unit vectors in R^n and suppose the projection of each of these vectors on the first k coordinates is of Euclidean norm at least ϵ. Show that $|X| \leq k/\epsilon^2$, and this is tight for all $\epsilon = k/2^r < 1$.