Problem Set # 6

Discrete Mathematics III (Probabilistic Method)
WS 2010/11

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The solutions are due on December 1st, 14:15pm.

You are welcome to submit **at most** two neatly written exercises each week. You must achieve a total of 15 full solutions during the semester.

You should try to solve all the exercises, they are part of the final exam.

1. Show that the random graph $G(n, p)$ is connected *a.a.s.*[1] if p is a bit larger than $\frac{\ln n}{n}$. Formally, prove that for every $\epsilon > 0$

$$\lim_{n\to\infty} Pr\left(G\left(n, (1+\epsilon)\frac{\ln n}{n}\right) \text{ is connected}\right) = 1.$$

2. Let $f(k)$ be the largest integer that **every** k-SAT formula[2] with $f(k)$ clauses is satisfiable. Determine $f(k)$.

3. $(*)$ Show that there is a finite n_0 such that any directed graph on $n > n_0$ vertices in which each outdegree is at least $\log_2 n - \frac{1}{10}\log_2\log_2 n$ contains an even simple directed cycle.

[1] A sequence of events E_n holds *a.a.s.*, that is, *asymptotically almost surely* if $\lim_{n\to\infty} Pr(E_n) = 1$. (cf an event E holding *a.s.* (*almost surely*), when $Pr(E) = 1$.)

[2] By a k-SAT formula we mean a boolean formula which is the conjunction of clauses, where a literal and its negation cannot appear in the same clause and each clause is the disjunction of *exactly* k distinct literals (i.e., a variable or its negation). For example $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_3 \vee \neg x_2 \vee x_5) \wedge (x_2 \vee \neg x_7 \vee x_1) \wedge (x_5 \vee x_4 \vee \neg x_1)$ is a 3-SAT formula. Note that in the standard terminology it is not required that there exactly k distinct literals.
The k-uniform hypergraph two-coloring problem (Property B) can be encoded as a k-SAT formula: for each edge $\{v_{i_1}, \ldots, v_{i_k}\}$ one can introduce two clauses, $(x_{i_1} \vee \ldots \vee x_{i_k})$ and $(\neg x_{i_1} \vee \ldots \vee \neg x_{i_k})$. Then a proper two-coloring corresponds to a satisfying assignment.