Problem Set # 7

Discrete Mathematics III (Probabilistic Method)
WS 2010/11

Tibor Szabó

The solutions are due on December 8th, 14:15pm.

You are welcome to submit **at most** two neatly written exercises each week. You must achieve a total of 15 full solutions during the semester.

You should try to solve all the exercises, they are part of the final exam.

1. Let X be a random variable taking integral nonnegative values, Prove that

$$Prob(X = 0) \leq \frac{Var(X)}{E(X^2)}.$$

2. Show that the random graph $G(n, p)$ is not connected *a.a.s.*, if the edge probability p is a bit smaller than $\frac{\ln n}{n}$. Formally, prove that for every $\epsilon > 0$

$$\lim_{n \to \infty} Pr\left(G\left(n, (1-\epsilon)\frac{\ln n}{n}\right) \text{ has an isolated vertex}\right) = 1.$$

Remark. This exercise, together with Problem 1 of the previous set shows that the property "connected" has a *sharp threshold* at $\frac{\ln n}{n}$. A function $f(n)$ is called a *threshold function* for the monotone property $\mathcal{P}$ if two things hold. On the one hand for $p \ll f(n)$ its is very unlikely that $G(n, p)$ has the property $\mathcal{P}$, on the other hand for $p \gg f(n)$ it is very likely that $G(n, p)$ possesses $\mathcal{P}$. Formally,

$$Pr(G(n, p) \text{ has property P}) \to \begin{cases} 0 & \text{if } p \ll f(n) \\ 1 & \text{if } p \gg f(n) \end{cases}$$

The threshold is *sharp* if for every $\epsilon > 0$ we have

$$Pr(G(n, p) \text{ has property P}) \to \begin{cases} 0 & \text{if } p < (1-\epsilon)f(n) \\ 1 & \text{if } p > (1+\epsilon)f(n) \end{cases}$$

3. (*) Show that there is a positive constant c such that the following holds. For any n reals $a_1, a_2, \ldots, a_n$ satisfying $\sum_{i=1}^n a_i^2 = 1$, if $(\epsilon_1, \ldots, \epsilon_n)$ is a $\{-1, 1\}$-random vector obtained by choosing each ϵ_i randomly and independently with uniform distribution to be either -1 or 1, then

$$Prob(|\sum_{i=1}^n \epsilon_i a_i| \leq 1) \geq c.$$