Problem Set # 8

Discrete Mathematics III (Probabilistic Method)
WS 2010/11

Tibor Szabó

The solutions are due on December 15th, 14:15pm.

You are welcome to submit **at most** two neatly written exercises each week. You must achieve a total of 15 full solutions during the semester.

You should try to solve all the exercises, they are part of the final exam.

1. Let X be a random variable with expectation $E(X) = 0$. Prove that for all $\lambda > 0$,
$$Prob[X \geq \lambda] \leq \frac{Var(X)}{Var(X) + \lambda^2}.$$

2. Let Y be the longest monotone subsequence in a random permutation of $[n]$. Show that there exists constants $c_1, c_2 > 0$ such that
$$Pr(c_1\sqrt{n} < Y < c_2\sqrt{n}) \quad \to 1$$

3. Let Z be the longest monotone increasing subsequence in a random permutation of $[n]$. Show that there exists constants $c_1, c_2 > 0$ such that
$$Pr(c_1\sqrt{n} < Z < c_2\sqrt{n}) \quad \to 1$$

4. Let $v_1 = (x_1, y_1), \ldots, v_n = (x_n, y_n)$ be n two dimensional vectors, where each x_i and each y_i is an integer whose absolute value does not exceed $\frac{2^{n/2}}{100\sqrt{n}}$. Show that there are two disjoint sets $I, J \subset \{1, 2, \ldots, n\}$ such that
$$\sum_{i \in I} v_i = \sum_{j \in J} v_j.$$

5. (*) Show that there is a positive constant c such that the following holds. For any n vectors $a_1, a_2, \ldots, a_n \in R^2$ satisfying $\sum_{i=1}^n ||a_i||^2 = 1$ and $||a_i|| \leq 1/10$, where $||\cdot||$ denotes the usual Euclidean norm, if $(\epsilon_1, \ldots, \epsilon_n)$ is a $\{-1, 1\}$-random vector obtained by choosing each ϵ_i randomly and independently with uniform distribution to be either -1 or 1, then
$$Prob(||\sum_{i=1}^n \epsilon_i a_i|| \leq 1/3) \geq c.$$