Problem Set # 9

Discrete Mathematics III (Probabilistic Method)
WS 2010/11

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The solutions are due on January 11th, 14:15pm.

You are welcome to submit **at most** two neatly written exercises each week. You must achieve a total of 15 full solutions during the semester.

You should try to solve all the exercises, they are part of the final exam.

1. Show that in the random graph $G(n, 1/2)$ the following holds a.a.s.: for any two disjoint subsets A and B of sizes $|A|, |B| \geq n/10$, we have that

$$\left| e(A, B) - \frac{|A||B|}{2} \right| = O\left(n^{3/2}\right),$$

   where $e(A, B)$ is the number of edges between A and B.

2. Show that in the random graph $G(n, p)$ all co-degrees are asymptotically $p^2 n$, provided p is large enough. Formally, show that for every $\epsilon > 0$ the following holds a.a.s.: for every two vertices $v, u \in V$, $||N(u) \cap N(v)| - p^2 n| \leq \epsilon p^2 n$.

   For how small p can you prove this? (the smaller, the better!)

3. Show that there is a graph on n vertices with minimum degree at least $n/2$ in which the size of every dominating set is at least $\Omega(\log n)$.

4. The presence of a k-clique in a graph G certainly implies that the chromatic number of G is at least k, but the converse of this statement is *very not true*; we proved in class that there exists triangle-free graphs with arbitrarily large chromatic number.

   Could a weaker statement be true? Formally: Is it necessary that if the chromatic number of a graph G is k, then G contains a topological $\frac{k}{2}$-clique? (i.e., H is called a topological s-clique if it consists of s *branch*vertices and a path between each of the $\binom{s}{2}$ pairs of branchvertices, such that these paths are internally pairwise vertex disjoint (and no vertex is an internal vertex of a path and an endpoint of another)).