bability Theory

athematical discipline concerned with the otion of probability and analysis of various random phenomena.

is part of mathematics, Probability Theory is PRECISE
(only its objects ~~is~~ are random~~ness~~, not the theory)

riginal motivation / goal:

To give a completely formal and precise mathematical model that describe various real-life phenomena,

These real-life phenomena can include ones that

- we consider "random" or
- we just think that randomness might be an effective way to study the phenomenon (whose nature we know to be deterministic but it is waaay too complicated/takes long time to study it in its full deterministic detail. Or whose nature we do not know.)

Real-life scenario with questions. Questions, where answer is unsure, unknown.	WANT →	Mathematical Model • precise • useful (we learn something about the answe

Benchmark Beispiele (Szenarien mit Zufall)

Glücksspiele: (Ursprung der Wahrscheinlichkeitstheorie)

1. Wir werfen ein Würfel. Was ist die Augenzahl?

2. Wir spielen Lotto. Wie viele richtige Zahlen haben wir?

3. Wir spielen ROT im Roulette 100 mals. Wie viel mal gewinnen wir?

4. Wir spielen ROT im Roulette bis wir zum ersten Mal gewinnen. Wie lang müssen wir warten?

Auch ein paar Beispiele, die NICHT Glücksspiele sind:

5. Wir telefonieren unseren besten Freund. Es ist besetzt. Wie lang müssen wir warten, bis er auflegt?

6. Wir besitzen eine Versicherungsfirma. Wie viele Schadensmeldungen erhalten wir zwischen 10^{00}–11^{00} am 5. Januar 2017?

---

1. – 6. sind verschiedene Zufallsexperimente

Wir möchten sinnvolle Modelle definieren, die auch hilfreich sind, etwas über die Zukunft zu lernen.

Common features of 1. - 6.

A There is a set Ω, where the result of the experiment lies in: 1. $\Omega = [6]$

> Notation: For $n \in \mathbb{N}$
> $[n] := \{1, 2, \dots, n\}$

2. $\Omega = \{0, 1, 2, 3, 4, 5, 6\}$

3. $\Omega = \{0\} \cup [100]$

4. $\Omega = \mathbb{N}$

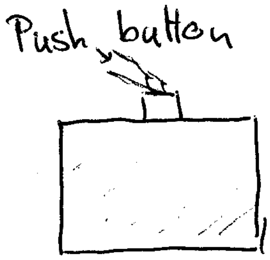
5. $\Omega = (0, +\infty)$

6. $\Omega = \{0\} \cup \mathbb{N}$

Elements of Ω are called elementary events

Ω is the set of elementary events

"Randomness automata"



$\longrightarrow \omega \in \Omega$ exactly one element of Ω comes out.

B.

- Sometimes we are interested in the occurance of an elementary event ("Does the dice show ⚅?")
- Sometimes we are interested in the occurance of a SET of elementary events. That is in a subset E of Ω.
  ("Does my friend talk more than 10 minutes?"
  "I'm not partikularly interested whether he talks exactly 4.17 minutes.")

Depending on the scenario there are particular subsets $E \subseteq \Omega$ of elementary events that I find "interesting". Such an $E \subseteq \Omega$ is called an <u>event</u>

C.

There are tendencies. If we push the button of the Box many times, we observe that for an event $E \subseteq \Omega$, there is a number $P(E) \in [0,1]$ such that roughly in $P(E)$-proportion of all these experiments the Box spits out a ω which is in E.

This number $P(E)$ we call the <u>probability of E</u>

Erste Hälfte der Vorlesung: Ω höchstens abzählbar
(d.h. $|\Omega| < \infty$ oder Ω abzählbar unendlich)

Definition: Ein diskreter Wahrscheinlichkeitsraum ist ein Paar $(\Omega, \mathbb{P})$ wobei

(DW1) Ω ist eine (höchstens abzählbare) Menge

(DW2) $\mathbb{P}: \Omega \to [0,1]$ ist eine Abbildung, sodass $\boxed{\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1}$

- Elemente von Ω: Elementarereignisse
- Ω: Menge der Elementarereignisse
- $\mathbb{P}$: Wahrscheinlichkeitsmaß
- $\mathbb{P}(\omega)$: Wahrscheinlichkeit des Elementarereignis $\omega \in \Omega$.
- Man kann $\mathbb{P}$ auf die ganze Potenzmenge $\mathcal{P}(\Omega)$ erweitern: $\forall E \in \mathcal{P}(\Omega)$ definieren wir

$$\boxed{\mathbb{P}(E) := \sum_{\omega \in E} \mathbb{P}(\omega)} \qquad \left(\text{Für } \omega \in \Omega: \ \mathbb{P}(\{\omega\}) = \mathbb{P}(\omega)\right)$$

- Eine Teilmenge $E \subseteq \Omega$ wird Ereignis genannt.
  $\Updownarrow$
  $(E \in \mathcal{P}(\Omega))$

Bemerkungen: • Da Ω höchstens abzählbar ist, kann man sich unter $\mathbb{P}$ eine Sequenz von Zahlen (alle zwischen 0 und 1) vorstellen.
• Oft wird Verteilung als Synonym für W-Maß verwendet.

Laplaceraum $|\Omega| < \infty$

$$\boxed{\forall \omega \in \Omega} \quad \boxed{p_\omega = \frac{1}{|\Omega|}}$$

Check: $\sum_{\omega \in \Omega} p_\omega = \sum_{\omega \in \Omega} \frac{1}{|\Omega|} = |\Omega| \frac{1}{|\Omega|} = 1$ ✓

Remark: • Does NOT exist when $|\Omega| = \infty$

• $\forall E \subseteq \Omega \qquad \mathbb{P}(E) = \frac{|E|}{|\Omega|} \qquad \left(= \frac{\text{"günstige"}}{\text{"mögliche"}} \right)$

Example ① ~~Wür~~ Dice: $\Omega = [6]$

when dice is fair: $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = \frac{1}{6}$

$$\mathbb{P}(\omega \text{ is even}) = p_2 + p_4 + p_6 = \frac{|\{2,4,6\}|}{|[6]|} = \frac{3}{6} = \frac{1}{2}$$

Important to find the $\boxed{\text{right}}$ model.
(a'la NOT every finite probability space is Laplace!)

Scenario: Two fair coins thrown. What is the probability of 1 Head?

Naive (false) answer of many: There are three possibilities 0, 1 or 2, SO(!) the probability is $\frac{1}{3}$.

How to convince one that this wrong?

• Experiment (Try 100 times) ~~...~~

• Convince about the RIGHT Laplaceraum in the background.

Paint coins (Red and Blue): Does the probability of seeing exactly one Head changes, just because of painting? NO

But now there are four options: B Head and R Head
, B Tail and R Head
, B Head and R Tail
, B Tail and R Tail

Is any one of them more likely than the other? NO

$\Rightarrow \Omega' = \{BH+RH, BT+RH, BH+RT, BT+RT\}$ $|\Omega'| = 4$

Laplace Raum $\forall \omega \in \Omega'$ $p_\omega = \frac{1}{4}$

~~Obs~~ The event of our interest $E = \{BH+RT, BT+RH\}$

$$P(E) = \frac{|E|}{|\Omega'|} = \frac{2}{4} = \frac{1}{2}$$ ✓

What happened?

- Our initial $\Omega = \{0,1,2\}$ had our event of interest (~~Obs~~ the 1) as an elementary event, but the space was NOT a Laplaceraum.
- We found a larger Model Ω', which was really a Laplaceraum and our event of interest ~~was~~ was an event $E \subseteq \Omega'$ in it.

Then we just counted $|\Omega'|$ $(=4)$

counted $|E|$ $(=2)$

and got the answer $\left(=\frac{2}{4}\right)$

So we need to know how to enumerate elements of (a set.)